ZIGGURATS AND ROTATION NUMBERS

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ABSTRACT. We establish the existence of new rigidity and rationality phenomena in the theory of nonabelian group actions on the circle, and introduce tools to translate questions about the existence of actions with prescribed dynamics into finite combinatorics. A special case of our theory gives a very short new proof of Naimi's theorem (i.e. the conjecture of Jankins–Neumann) which was the last step in the classification of taut foliations of Seifert fibered spaces.

CONTENTS

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1. Introduction

This paper introduces new techniques and uncovers a range of new phenomena in the topological theory of nonabelian group actions on the circle. We develop computational tools which let us reduce subtle dynamical questions to finite combinatorics, and then connect these questions back to (hyperbolic) geometry and low-dimensional topology.

1.1. **A character variety for** $\mathrm{Homeo}^+(S^1)$**.** Linear actions of finitely generated groups are parameterized by *character varieties*. Because a character variety is an algebraic variety, the characters (which capture the abstract dynamics of a representation) are *polynomial functions*, and this makes answering questions about the existence or nonexistence of actions with certain properties *computationally tractable*.

It is natural to want to generalize this theory to *nonlinear* actions of finitely generated groups on manifolds. In general there is probably no chance of such a generalization; however, for the case of *group actions on circles*, we believe some elements of the theory can be developed.

Associated to a linear representation $\rho : \Gamma \to G$ is a trace $\chi : \Gamma \to \mathbb{R}$. Given Γ and G, a trace is determined by its values on finitely many $g \in \Gamma$, and for each $g \in \Gamma$ the value $\chi(g)$ is an invariant of the conjugacy class of $\rho(g)$. Associated to a nonlinear representation $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ one has the *rotation number* $\mathrm{rot} : \Gamma \to \mathbb{R}/\mathbb{Z}$, and for each $g \in \Gamma$ the value $\mathrm{rot}(g)$ is a (complete) invariant of the *semi-conjugacy* class of $\rho(g)$ (i.e. the equivalence relation generated by dynamical semi-conjugacy).

Given a group G and a collection of elements g_i and h in G, we would like to know what possible dynamics can be achieved by h under representations of G into $\mathrm{Homeo}^+(S^1)$ for which the g_i have prescribed dynamics, either in the form of a constraint on the rotation numbers of the g_i, or by imposing the condition that the g_i are conjugate to rotations through prescribed angles. A good theory should make it possible to *compute* these values. In fact, it is one of the main goals of this paper to develop tools to allow one to describe the shadows (i.e the projections under rot) of such "algebraic" subsets of $\mathrm{Hom}(G, \mathrm{Homeo}^+(S^1))$, at least for G free, or a free product of cyclic groups.

One significant point of difference between characters and rotation numbers is that it is *not* true that knowing the rotation numbers of finitely many elements suffices to determine the rest. The following example depends on knowledge of some elementary facts about hyperbolic structures on surfaces; see e.g. [14] for a reference.

Example 1.1. Consider a family of representations of F_2 into $\mathrm{PSL}(2, \mathbb{R})$ (which acts on $\mathbb{RP}^1$) associated to a family of (incomplete) hyperbolic structures on a once-punctured torus. We identify F_2 with the fundamental group of this torus, so that the generators a and b correspond to the meridian and longitude, and the commutator $[a, b]$ corresponds to the puncture.

At the complete finite area structure, every element is hyperbolic or parabolic, and therefore every element has rotation number 0 (mod $\mathbb{Z}$). As we deform the image of $[a, b]$ from a parabolic to an elliptic element, $\mathrm{rot}([a, b])$ becomes nonzero. However, for every finite collection of elements not conjugate to a power of $[a, b]$, the image stays hyperbolic, and the rotation number stays zero, for sufficiently

small perturbations. Moreover, there is a 2 parameter family of deformations of the hyperbolic structure keeping the cone structure near the puncture fixed, and under such a deformation some elements switch from hyperbolic to elliptic or back.

On the other hand, there is a certain sense in which knowledge of *all* rotation numbers together with certain homological data *does* determine a representation, again up to the relation of semi-conjugacy. The precise statement is due to Ghys [9] and is further refined by Matsumoto [15], and can be expressed in terms of the "Euler class in integral bounded cohomology"; since a proper discussion of this would take us too far afield, we do not pursue it here.

If it is not true that finitely many rotation numbers determine the rest, it is nevertheless true that the rotation numbers of (say) the generators strongly *constrain* the rotation numbers of the other elements, and in quite an interesting way. The main problem on which we focus in this paper is therefore the following:

Motivating Question. *Given a free group F, and an element w of F, and given values of the rotation numbers of the generators, what is the set of possible rotation numbers of w?*

We are especially focussed on the special case that w is a *positive* word in the generators of F; i.e. it is contained in the *semigroup* they generate. In this case the theory simplifies significantly, and we are able to obtain strong results.

1.2. **Phase locking and greedy rationals.** One of the best known and best studied examples of nonlinear phase locking in dynamics is the phenomenon of *Arnol'd tongues*: the introduction of nonlinear noise into a family of circle homeomorphisms tends to produce *periodic orbits*; informally, we call this the phenomenon of *greedy rationals*. In this paper we are concerned with another manifestation of this phenomenon, in the context of *nonabelian group dynamics*. We find strong constraints on the dynamics of free group actions on the circle that maximize certain dynamical quantities, and in many case show that these these constraints are powerful enough to guarantee periodic orbits and therefore rational rotation numbers.

1.3. **New proofs of old theorems.** The classification of taut foliations of Seifert fibered 3-manifolds was the culmination of the work of many people, including Thurston, Brittenham, Eisenbud, Hirsch, Jankins, Neumann and others, and was completed by Naimi [17] by proving an outstanding conjecture of Jankins–Neumann [12]. The conjecture of Jankins–Neumann is equivalent to an analysis of the first non-trivial example in our theory, and our methods lead to a new and very short proof of Naimi's result, thereby embedding this classical work into a new and more powerful context.

1.4. **Statement of results.** For simplicity, the results in our paper are proved for actions of a free group of rank 2 on the circle. However, the generalization to arbitrary rank is straightforward. Statements of the main theorems for arbitrary rank can be found in the appendix; the proofs are routine generalizations of the proofs given in the paper for the rank 2 case, and are left to the reader.

§ 2 is devoted to setting up notation and proving some elementary results.

We introduce the following notation. Let F be a free group of rank 2 generated by elements a, b. Let $\mathrm{Homeo}^+(S^1)^\sim$ denote the universal central extension of the group

of orientation-preserving homeomorphisms of the circle. For $\varphi \in \mathrm{Homeo}^+(S^1)^\sim$, let $\mathrm{rot}^\sim(\varphi)$ denote the (real-valued) rotation number.

Given $w \in F$ and real numbers r, s, let $X(w, r, s)$ denote the set of values of $\mathrm{rot}^\sim(w)$ achieved by representations $F \to \mathrm{Homeo}^+(S^1)^\sim$ for which $\mathrm{rot}^\sim(a) = r$ and $\mathrm{rot}^\sim(b) = s$, and let $R(w, r, s)$ denote the supremum of the set $X(w, r, s)$. In words, $R(w, r, s)$ is the supremum of $\mathrm{rot}^\sim(w)$ under all representations for which $\mathrm{rot}^\sim(a) = r$ and $\mathrm{rot}^\sim(b) = s$. It turns out (see Lemma 2.13 and Proposition 2.16) that $X(w, r, s)$ is a *compact* interval with maximum $R(w, r, s)$ (so that the supremum is really *achieved*) and minimum $-R(w, -r, -s)$, so we focus almost exclusively on $R(w, r, s)$ throughout the paper.

A word $w \in F$ is *positive* if it does not contain a^{-1} or b^{-1}. Positive words are the focus of § 3, and our strongest results are stated and proved for positive words.

The first main result we prove is the Rationality Theorem:

Rationality Theorem 3.2. *Suppose w is positive. If r and s are rational, so is $R(w, r, s)$. Moreover, if w is not a power of a or b, the denominator of $R(w, r, s)$ is no bigger than the minimum of the denominators of r and of s.*

Moreover, we are able to reduce the computation of $R(w, r, s)$ in any case to a *finite* (albeit complicated) combinatorial problem, which can be solved explicitly, e.g. by computer.

Complementing the rationality theorem is the Stability Theorem:

Stability Theorem 3.7. *Suppose w is positive. Then R is locally constant from the right at rational points; i.e. for every pair of rational numbers r and s, there is an $\epsilon(r, s) > 0$ so that $R(w, \cdot, \cdot)$ is constant on $[r, r + \epsilon) \times [s, s + \epsilon)$.*

Conversely, if $R(w, r, s) = p/q$ (where p/q is reduced) and the biggest power of consecutive a's in w (resp. b's) is a^m (resp. b^n), then $R(w, r + 1/mq, s) \geq p/q + 1/q^2$ (resp. $R(w, r, s + 1/nq) \geq p/q + 1/q^2$).

We think of this theorem as a nonabelian group theoretic cousin of the phenomenon of *Arnol'd tongues*, with the "tongues" corresponding to the regions in the r–s plane where $R(w, \cdot, \cdot)$ is locally constant. The Stability Theorem shows that the sizes of these tongues decrease with q, but experiments suggest a sharper relationship $\epsilon \sim q^{-1}$. This seems very exciting to us, and deserves further exploration.

For the special case of $w = ab$, the combinatorics becomes simple enough to allow a complete solution. This leads to an elementary proof of the following theorem, first conjectured by Jankins–Neumann and proved by Naimi:

$w = ab$ Theorem 3.9. *For $0 \leq r, s < 1$ there is an equality*

$$R(ab, r, s) = \sup_{p_1/q \leq r, \, p_2/q \leq s} (p_1 + p_2 + 1)/q$$

As remarked earlier, this theorem is the last step in the classification of taut foliations of Seifert fibered spaces.

If we fix w and a rational number $p/q = \mathrm{rot}^\sim(a)$, the function $R(w, p/q, \cdot)$ is nondecreasing and takes on a discrete set of values (however, it is typically discontinuous). The next theorem says that this function is continuous from the right, and the points of discontinuity are rational, and can be determined by a finite procedure.

Stairstep Theorem 3.11. *Let w be positive, and suppose we are given rational numbers p/q and c/d so that c/d is a value of $R(w, p/q, \cdot)$ (so necessarily $d \leq q$). Then $\inf\{t : R(w, p/q, t) = c/d\}$ is rational, and there is an algorithm to compute it. Moreover, if u/v is this infimal value, $R(w, p/q, u/v) = c/d$.*

Although we are not able to give a precise description of the level sets of R, we are able to give a description of the frontier of the set where $R(w, \cdot, \cdot) \geq p/q$ for a given rational number p/q. This is summarized in the following Isobar Theorem:

Isobar Theorem 3.32. *Let w be positive. For any rational p/q the set of $r, s \subset [0, 1] \times [0, 1]$ such that $R(w, r, s) \geq p/q$ is a finite sided rational polyhedron, whose boundary consists of finitely many horizontal or vertical segments.*

One subtle point is that the set where $R(w, r, s) = p/q$ is *not* in general a finite sided rational polyhedron, and can in fact be extremely complicated.

§ 4 moves on to the case of arbitrary w. The Rationality Theorem generalizes to words w which are *semipositive*; i.e. that contain to a^{-1} or no b^{-1} (Theorem 4.2).

We conjecture (Conjecture 4.5) that for arbitrary w and rational r, s, $R(w, r, s)$ is rational. Given w, a compactness argument (using circular orders rather than actions) shows that there is a representation with $\mathrm{rot}^\sim(a) = r$ and $\mathrm{rot}^\sim(b) = s$ realizing $\mathrm{rot}^\sim(w) = R(w, r, s)$. Counterfactually, we suppose that $\mathrm{rot}^\sim(w)$ is irrational, and then try to modify the dynamics to increase it. This gives rise to a dynamical problem of independent interest called the *interval game*. Roughly speaking, given a finite collection of homeomorphisms $\varphi_1, \varphi_2, \cdots, \varphi_m$ and an irrational number θ, we would like to find an interval I and an integer n for which $R_\theta^i(I^+)$ is not contained in $\varphi_j(I)$ for any $0 \leq i \leq n$ and any j, but $R_\theta^n(I^+) \in I$. Here I^+ denotes the rightmost point of the interval I, and R_θ is the rotation $R_\theta : z \to z + \theta$.

We show (Theorem 4.14 and Theorem 4.15) that the interval game can be solved for *generic* C^1 diffeomorphisms φ_i, and for *all* cases of a single homeomorphism φ except when φ is itself a rotation R_ϕ. When φ is a rotation, the set of pairs (θ, ϕ) for which the interval game can be won turns out to be an interesting open, dense subset of the unit square, which is described in Theorem 4.16. It follows that for w containing a single a^{-1}, there are strong constraints on the possible irrational values of $R(w, r, s)$, in terms of w and s, namely that they are contained in the complement of an explicit dense G_δ.

We are also concerned throughout this paper with representations satisfying the stronger constraint for which a and b are conjugate to rotations R_r and R_s respectively. We denote by $R(w, r-, s-)$ the supremum of $\mathrm{rot}^\sim(w)$ over representations of this kind. It turns out that for w positive, $R(w, r-, s-)$ is the limit of $R(w, r', s')$ as $r' \to r$ and $s' \to s$ from below. Because of this, we can calculate $R(w, r-, s-)$ to any degree of accuracy, and in many (most) cases, show that it is rational. By a trick, for rational r, s and any word w there is some *positive* word $w'(w, r, s)$ and an explicit integer $N(w, r, s)$ for which $R(w, r-, s-) = N + R(w', r-, s-)$, and therefore we can (numerically) compute $R(w, r-, s-)$ for rational r, s and for *arbitrary* words w. This achieves our goal of making the theory computationally tractable.

2. Basic definitions and properties

In the sequel we fix F, a free group on two generators a and b. We denote by $\mathrm{Homeo}^+(S^1)$ the group of orientation-preserving homeomorphisms of S^1, and by

$\mathrm{Homeo}^+(S^1)^\sim$ the group of orientation-preserving homeomorphisms of $\mathbb{R}$ covering homeomorphisms of S^1 under the covering projection $\mathbb{R} \to S^1$. There is a central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}^+(S^1)^\sim \to \mathrm{Homeo}^+(S^1) \to 0$$

and the kernel of $\mathrm{Homeo}^+(S^1)^\sim \to \mathrm{Homeo}^+(S^1)$ is the group of integer translations of $\mathbb{R}$.

We denote by $R_\theta \in \mathrm{Homeo}^+(S^1)$ the homeomorphism $z \to z + \theta$; we call R_θ the *rigid rotation* through angle θ. By abuse of notation, we also denote by R_θ its lift to $\mathrm{Homeo}^+(S^1)^\sim$. In the first case, θ should be thought of as an element of S^1; in the second, as an element of $\mathbb{R}$. Which meaning is intended should be clear from context.

2.1. **Rotation number.** We assume the reader is familiar with Poincaré's rotation number; for a basic reference, see e.g. Ghys [10] or Herman [11].

Rotation number is a function $\mathrm{rot} : \mathrm{Homeo}^+(S^1) \to \mathbb{R}/\mathbb{Z}$ which admits a real valued lift $\mathrm{rot}^\sim : \mathrm{Homeo}^+(S^1)^\sim \to \mathbb{R}$. The following proposition summarizes some basic properties of these functions. These properties are all well-known, and we will use them without comment throughout the paper.

Proposition 2.1. *Rotation number satisfies the following properties.*

(1) rot *and* $\mathrm{rot}^\sim$ *are conjugacy invariants, and are continuous in the C^0 topology.*

(2) *A homeomorphism $\varphi \in \mathrm{Homeo}^+(S^1)$ has rational rotation number with denominator q if and only if it has a periodic orbit with period q.*

(3) rot *and* $\mathrm{rot}^\sim$ *are homomorphisms when restricted to any cyclic (in fact, amenable) group of homeomorphisms.*

(4) *For any $\varphi \in \mathrm{Homeo}^+(S^1)^\sim$ with $\mathrm{rot}^\sim(\varphi) = r$, the map $t \to \mathrm{rot}^\sim(R_t \circ \varphi)$ defines a continuous, nondecreasing, surjective map from $[0,1] \to [r, r+1]$ (see [11]).*

Some more substantial, but nevertheless classical results, concern when a homeomorphism is conjugate to some R_θ:

Theorem 2.2 (Denjoy (see [11]))**.** *If φ is C^2 and has irrational rotation number, φ is (C^0) conjugate to a rigid rotation with the same rotation number.*

Theorem 2.3 (Herman, Yoccoz [11, 22])**.** *There is a subset of the set of irrational numbers of full measure, so that if θ is an element of this set, any φ which is C^∞ and is conjugate to R_θ, is conjugate by a C^∞ diffeomorphism.*

For concreteness, we call an irrational θ as in Herman–Yoccoz theorem a *Herman number*.

2.2. **Extremal representations.** The fundamental question we are concerned with in this paper is the following:

Question 2.4. *Given an element $w \in F$ and real numbers r, s, what is the supremum of $\mathrm{rot}^\sim(\rho(w))$ over all representations $\rho : F \to \mathrm{Homeo}^+(S^1)^\sim$ for which $\mathrm{rot}^\sim(\rho(a)) = r$ and $\mathrm{rot}^\sim(\rho(b)) = s$?*

The content of Jankins–Neumann's paper [12] is a partial analysis of this question for the important but very special case $w = ab$. They formulated a complete

conjectural answer, and went a long way towards proving it; their conjecture was subsequently established by Naimi [17].

We discuss this special case in § 3.3 and give a new, much shorter, and more direct proof of Jankins–Neumann's conjecture (see Theorem 3.9).

Remark 2.5. One can ask variations on this question for more complicated groups G and elements $w \in G$; some results in this direction are obtained in [2]. The techniques and results obtained in this paper generalize in a straightforward way to the case of a free group F of rank > 2.

We introduce the following notation.

Notation 2.6. Given $w \in F$ and $r, s \in \mathbb{R}$, we let $R(w, r, s)$ denote the supremum of $\mathrm{rot}^\sim(\rho(w))$ over all $\rho : F \to \mathrm{Homeo}^+(S^1)^\sim$ for which $\mathrm{rot}^\sim(\rho(a)) = r$ and $\mathrm{rot}^\sim(\rho(b)) = s$.

Notation 2.7. Let $h_a, h_b : F \to \mathbb{Z}$ denote the homomorphism which "counts" the (algebraic) number of a's or b's in $w \in F$; i.e. the homomorphisms defined on generators by $h_a(a) = 1$, $h_a(b) = 0$ and $h_b(a) = 0$, $h_b(b) = 1$ respectively.

Lemma 2.8. *Let* $\rho : F \to \mathrm{Homeo}^+(S^1)$ *be any representation, and let* $[r] = \mathrm{rot}(a)$, $[s] = \mathrm{rot}(b)$ *for some* $[r], [s] \in \mathbb{R}/\mathbb{Z}$. *If* r, s *are any lifts of* $[r], [s]$ *to* $\mathbb{R}$, *there is a unique lift* $\rho^\mathbb{Z} : F \to \mathrm{Homeo}^+(S^1)^\sim$ *with* $\mathrm{rot}^\sim(a) = r$, $\mathrm{rot}^\sim(b) = s$.

Proof. Pick any lift, then multiply the generators by suitable powers of the center. □

The function R has some elementary properties, which we record.

Lemma 2.9 (periodicity)**.** *For any integers* n, m *we have* $R(w, r + n, s + m) = R(w, r, s) + nh_a(w) + mh_b(w)$.

Proof. Multiply generators by suitable powers of the center. □

Lemma 2.10 (left-right symmetry)**.** *If* $\overline{w}$ *denotes the string obtained by reversing the order of* w, *then* $R(w, r, s) = R(\overline{w}, r, s)$.

Proof. Changing the orientation of the circle multiplies rotation numbers by -1. Taking inverses multiplies rotation numbers by -1. The composition of these two operations proves the lemma. □

There are *a priori* estimates on $R(w, r, s)$ in terms of the combinatorics of w.

Lemma 2.11. *Suppose* w *is conjugate into the form* $w = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_m} b^{\beta_m}$ *for some* $m = m(w)$. *Then* $R(w, r, s) \le m(w) + rh_a(w) + sh_b(w)$.

Proof. The proof is elementary, given some facts from the theory of stable commutator length (denoted scl, see Definition 3.22); we will return to this connection in § 3.8, and the reader can consult this section or [5] for definitions and details.

The chain $w - a^{h_a} - b^{h_b}$ is homologically trivial, and its stable commutator length is easily shown to satisfy the inequality $\mathrm{scl}(w - a^{h_a} - b^{h_b}) \le m(w)/2$. On the other hand, the function $\mathrm{rot}^\sim$ is a homogeneous quasimorphism with defect at most 1 (this is a restatement of the Milnor–Wood inequality in the language of quasimorphisms), so the inequality follows from generalized Bavard duality ([5], Thm. 2.79). □

Remark 2.12. Evidently Lemma 2.11 is implied by the stronger inequality

$$R(w, r, s) \leq 2 \cdot \mathrm{scl}(w - a^{h_a} - b^{h_b}) + r h_a(w) + s h_b(w)$$

If r and s are rational, so is the right hand side of this inequality.

Lemma 2.13. *For any w, r, s the maximum $R(w, r, s)$ is achieved on some representation.*

Proof. Rotation number depends (continuously) only on the circular order derived by the action of the group on any orbit. The space of such circular orders is compact, and so is the subset for which a and b have a specific rotation number (mod $\mathbb{Z}$). By Lemma 2.8 and compactness, $\mathrm{rot}^{\sim}(w)$ is maximized on some circular order (which can be realized in many ways by an action). $\square$

Lemma 2.14. *For fixed w, the function $R(w, r, s)$ is lower semi-continuous in r and s.*

Proof. Lower semi-continuity means that the value of a limit is a least as big as the limit of the values. Let $r_i, s_i \to r, s$ and let ρ_i be representations for which the maximum is achieved. Some subsequence of the ρ_i converges up to semi-conjugacy to a limit ρ, and $\mathrm{rot}^{\sim}(\rho(w)) = \lim_i \mathrm{rot}^{\sim}(\rho_i(w))$. $\square$

Notation 2.15. Given $w \in F$ and $r, s \in \mathbb{R}$, we let $X(w, r, s)$ denote the set of values of $\mathrm{rot}^{\sim}(\rho(w))$ over all $\rho : F \to \mathrm{Homeo}^+(S^1)^{\sim}$ for which $\mathrm{rot}^{\sim}(\rho(a)) = r$ and $\mathrm{rot}^{\sim}(\rho(b)) = s$.

Note that $h_a(w)r + h_b(w)s \in X(w, r, s)$ for all w, r, s. The following proposition shows that $X(w, \cdot, \cdot)$ can be recovered from $R(w, \cdot, \cdot)$.

Proposition 2.16. *For any w and fixed r, s, the set $X(w, r, s)$ is a compact interval, with maximum $R(w, r, s)$ and minimum $-R(w, -r, -s)$.*

Proof. The set of representations for which the rotation numbers of a and b are prescribed is path-connected. This implies that $X(w, r, s)$ is connected, and therefore an interval. The maximum is $R(w, r, s)$ by definition, and the minimum is $-R(w, -r, -s)$ because changing the orientation of the circle negates rotation number. $\square$

2.3. **Approximation by smooth diffeomorphisms.** For later use, we state and prove some elementary facts about approximating homeomorphisms by diffeomorphisms.

Lemma 2.17. *Any $\varphi \in \mathrm{Homeo}^+(S^1)^{\sim}$ can be C^0 approximated by a C^∞ diffeomorphism $\varphi' \in \mathrm{Homeo}^+(S^1)^{\sim}$ with the same rotation number.*

Proof. First of all, it is obvious that φ can be C^0 approximated by C^∞ diffeomorphisms φ^+ and φ^- in such a way that $|\varphi(p) - \varphi^{\pm}(p)| < \epsilon$ for all p, while $\varphi^+(p)$ is to the right of $\varphi(p)$ and $\varphi^-(p)$ is to the left of $\varphi(p)$. One way to do this is to approximate the graph of φ by smooth graphs from above and below. Evidently $\mathrm{rot}^{\sim}(\varphi^-) \leq \mathrm{rot}^{\sim}(\varphi) \leq \mathrm{rot}^{\sim}(\varphi^+)$.

Now for $t \in [0, 1]$ define $\varphi_t(p) = t\varphi^-(p) + (1 - t)\varphi^+(p)$. This defines a smooth family interpolating between φ^- and φ^+, so some element of this family has the same rotation number as φ. Moreover, every element of this family satisfies $|\varphi_t(p) - \varphi(p)| < \epsilon$ for all p. $\square$

Lemma 2.18. *If both r and s are either rational or Herman numbers, every θ in the interior of $X(w, r, s)$ is realized by a smooth representation. If both r, s are Herman numbers, every θ in the interior of $X(w, r, s)$ is realized by a representation in which a and b are smoothly conjugate to R_r and R_s respectively.*

Proof. By Lemma 2.17, there are smooth representations with $\mathrm{rot}^\sim(a) = r$ and $\mathrm{rot}^\sim(b) = s$ and for which $\mathrm{rot}^\sim(w)$ is as close to any θ in $X(w, r, s)$ as we like; hence we can certainly realize a *dense* subset of θ in $X(w, r, s)$ by smooth representations. On the other hand, the set of diffeomorphisms whose rotation number is equal to a fixed rational or Herman number is path connected.

Indeed, if φ and φ' are both smooth with the same rational rotation number, they both have a finite orbit with the same dynamics. We can easily find a 1-parameter family φ_t interpolating between them with the same finite orbit, and hence the same rotation number.

If φ and φ' are both smooth with the same irrational Herman rotation number r, they are both smoothly conjugate to R_r. The two conjugating maps can be joined by a smooth path of conjugating maps, exhibiting a 1-parameter family φ_t all with rotation number r (in fact, all conjugate to R_r) interpolating between them.

In conclusion, the set of $\theta \in X(w, r, s)$ realized by smooth representations is dense and connected. The lemma follows. □

Remark 2.19. Shigenori Matsumoto communicated the following short proof that for any r and s (not necessarily rational or Herman irrational), every θ in the interior of $X(w, r, s)$ is realized by a smooth representation. This follows by showing that the space of C^∞ diffeomorphisms with a given rotation number r is pathwise connected. For rational r this is obvious; for irrational r, let f be any smooth diffeomorphism with $\mathrm{rot}(f) = r$, and define

$$f_{t,\theta} = R_\theta \circ ((1 - t)f + t\,\mathrm{id})$$

Because r is irrational, for each t there is a *unique* $\theta(t)$ so that $\mathrm{rot}(f_{t,\theta(t)}) = r$; see e.g. [11]. Since the subset $\{(t, \theta(t))\}$ is closed in $[0, 1] \times S^1$ (being the preimage of r under the continuous function rot) it follows that $t \to \theta(t)$ is continuous.

3. Positive words

3.1. **Positivity and rationality.**

Definition 3.1. A (necessarily reduced) word w in F is *positive* if it contains no a^{-1} or b^{-1} (i.e. it contains only a's and b's).

In this section we adhere to the convention that w is *positive* unless we explicitly say otherwise. In this case we are able to obtain complete and precise answers, and a surprisingly rich structure theory.

The first surprise is the following Rationality Theorem, which says that for r, s rational and w positive, $R(w, r, s)$ is rational, and there is an *a priori* bound on its denominator. In fact, it turns out that the computation of $R(w, r, s)$ in any given case can be reduced to a finite combinatorial question!

Theorem 3.2 (Rationality Theorem)**.** *Suppose w is positive. If r and s are rational, so is $R(w, r, s)$. Moreover, if w is not a power of a or b, the denominator of $R(w, r, s)$ is no bigger than the minimum of the denominators of r and of s.*

Proof. Consider any action of F on S^1 for which $\mathrm{rot}^\sim(a) = r$ and $\mathrm{rot}^\sim(b) = s$. We will show how to modify this action without decreasing $\mathrm{rot}^\sim(w)$, until w has a periodic orbit with period no bigger than that of a or b. In fact, the new w will have a periodic orbit which can be taken to biject (in a natural way) with a subset of a periodic orbit of either a or b. This will prove the proposition. We call this method of starting with any representation, and modifying the dynamics without decreasing $\mathrm{rot}^\sim(w)$ until w has some desired property, the *method of perturbation*.

Since r and s are rational, both a and b have finite orbits. If $r = p_1/q_1$ then there are points x_i for $0 \le i \le q_1 - 1$ for which $a(x_i) = x_{i+p_1}$ (indices taken mod q_1); similarly there are y_j for $0 \le j \le q_2 - 1$ for which $b(y_j) = y_{j+p_2}$ (indices mod q_2). Denote the union of the x_i by Σ_x, the union of the y_j by Σ_y, and the union of both by Σ.

We can modify the dynamics of a on the complement of Σ_x (and similarly for b) without changing their rotation numbers. Replacing a by some new a' with the property that $a'(p) \ge a(p)$ for all p cannot decrease $\mathrm{rot}^\sim(w)$, since only positive powers of a appear in w; a similar statement holds for b.

Define maps α^+ and β^+ by $\alpha^+(p) = a(x_{i+1})$ for $p \in (x_i, x_{i+1}]$ and $\beta^+(p) = b(y_{j+1})$ for $p \in (y_j, y_{i+1}]$. Note that although these maps are not homeomorphisms, they are monotone and therefore still have a well-defined rotation number.

Define w^+ to be the composition obtained by replacing a and b by α^+ and β^+ in the word w. Evidently $\mathrm{rot}^\sim(w^+) \ge \mathrm{rot}^\sim(w)$. On the other hand, for any ϵ we can choose a', b' so that $\alpha^+(p) - a'(p) \le \epsilon$ whenever $p - x_i > \epsilon$ (if $p \in (x_i, x_{i+1}]$) and similarly for b' and y_j. Set ϵ less than half the distance between distinct elements of Σ. Then there is some initial choice of p so that $|(w^+)^n(p) - w^n(p)| < \epsilon$ for any integer n, and therefore $\mathrm{rot}^\sim(w^+) = \mathrm{rot}^\sim(w)$.

It remains to estimate the denominator of $\mathrm{rot}^\sim(w^+)$. Since w is by hypothesis not a power of a, some conjugate of w ends with a, and therefore the corresponding conjugate of w^+ takes Σ_x into itself. The denominator of $\mathrm{rot}^\sim(w^+)$ is the least period of an orbit, and is therefore $\le q_1$. Interchanging a and b gives the desired estimate. $\square$

Note that Theorem 3.2 gives an *algorithm* to compute $R(w, r, s)$ for positive w and rational r and s. For each configuration of Σ in the circle, the rotation number can be read off from the map w^+ from Σ to itself. This rotation number only depends on the relative order of the points in Σ; there are only finitely many possible configurations, so by examining each of them in turn we can compute the maximum. This is pursued more systematically in § 3.2.

Lemma 3.3. *Suppose w is positive and fixed. Then $R(w, r, s)$ is monotone nondecreasing in r and in s.*

Proof. Let ρ be a representation maximizing $\mathrm{rot}^\sim(w)$ for fixed r, s. We may increase r by replacing a with the composition $R_t \circ a$ where R_t is a rotation through angle t. By Proposition 2.1 bullet (4) we can prescribe $\mathrm{rot}^\sim(R_t \circ a) \ge r$.

Since w is positive, replacing a with $R_t \circ a$ cannot decrease $\mathrm{rot}^\sim(w)$. $\square$

Lemma 3.4. *Suppose w is positive. If $r = p/q$ is rational, then $R(w, r, s)$ is rational with denominator $\le q$ for all s.*

Proof. As s increases in some interval, $R(w, r, s)$ is nondecreasing, by Lemma 3.3. On the other hand, it is rational with denominator $\leq q$ for all rational s by Theorem 3.2, and therefore by lower semicontinuity (i.e. Lemma 2.14) it is rational with denominator $\leq q$ for all s. □

3.2. *XY* **words and dynamics.** We now study the combinatorics of Σ_x and Σ_y (recall the notation and setup from the proof of Theorem 3.2). *A priori* it might be possible that $R(w, p_1/q_1, p_2/q_2)$ can only be achieved by some configuration where $\Sigma_x \cap \Sigma_y$ is nonempty. However, the following lemma shows we do not need to consider such configurations.

Lemma 3.5. *Suppose w is positive. $R(w, p_1/q_1, p_2/q_2)$ is achieved for some configuration where Σ_x and Σ_y are disjoint.*

Proof. Suppose we have a configuration in which $x_i = y_j$ for some i, j (i.e. in which Σ_x and Σ_y are not disjoint). Perturb x_i slightly so that $y_{j-1} < x_i < y_j$. We apply one of α^+ or β^+ in turn and see how our new orbit compares to the old one. After applying some power of α^+ we might end up at x_i. But if we then apply β^+ then because $\beta^+(x_i) = \beta^+(y_j)$ the new orbit immediately catches up to the old.

Conversely, if we apply some power of β^+ and end up at y_j and then apply α^+, because $\alpha^+(y_j) = \alpha^+(x_{i+1})$ the new orbit pulls ahead of the old. In either case, we definitely do not lag, and the rotation number is no smaller in the new configuration. □

Because of Lemma 3.5, in the sequel we will assume that Σ_x and Σ_y are disjoint, and therefore $|\Sigma| = q_1 + q_2$. The configuration of Σ_x and Σ_y in S^1 can be encoded (up to conjugacy) by a cyclic word W in letters X and Y containing q_1 X's and q_2 Y's; call such a word *admissible* for the pair q_1, q_2. The number of cyclic words admissible for q_1, q_2 is exponential in $\min(q_1, q_2)$, but for fixed q_1 (say), is polynomial in q_2.

There is a dynamical system, generated by transformations a and b, whose orbit space is the letters of W, as follows. The element a acts by moving to the right until we read off $p_1 + 1$ X's, counting the X we start on, if we start on an X. The element b acts by moving to the right until we read off $p_2 + 1$ Y's, counting the Y we start on, if we start on a Y. A maximal consecutive string a^m "hops" over $mp_1 + 1$ X's, and a maximal consecutive string b^n "hops" over $np_2 + 1$ Y's; we refer to these prosaically as *a-hops* and *b-hops* in the sequel. Since a and b are monotone with respect to the cyclic order, it makes sense to define the ($\mathbb{Q}$-valued) rotation number $R(w, p_1/q_1, p_2/q_2, W)$ for any admissible q_1, q_2 cyclic word W.

Lemma 3.5 implies $R(w, p_1/q_1, p_2/q_2) = \max_W R(w, p_1/q_1, p_2/q_2, W)$. For convenience we also define $r(w, p_1/q_1, p_2/q_2) := \min_W R(w, p_1/q_1, p_2/q_2, W)$ for reduced $p_1/q_1, p_2/q_2$.

Example 3.6. We compute $R(ab, 2/3, 1/2)$. Up to cyclic permutation, there are 2 admissible $3, 2$ words, namely $XXXYY$ and $XXYXY$. Each application of a skips over 3 X's, and each application of b skips over 2 Y's. We apply the letters of w from right to left to compute R; therefore we think of our group acting on S^1 on the left.

For the case $XXXYY$, there is an orbit of period 1:

$$XX\bar{X}YY \xrightarrow{b} XXXY\bar{Y} \xrightarrow{a} XX\bar{X}YY$$

and the rotation number is 1.

For the case $XXYXY$, there is an orbit of period 2:

$$XXY\bar{X}Y \xrightarrow{b} XX\bar{Y}XY \xrightarrow{a} X\bar{X}YXY \xrightarrow{b} XXYX\bar{Y} \xrightarrow{a} XXY\bar{X}Y$$

and the rotation number is $3/2$. Hence $R(ab, 2/3, 1/2) = 3/2$.

Complementing the Rationality Theorem is the following *Stability Theorem*:

Theorem 3.7 (Stability Theorem). *Suppose w is positive. Then R is locally constant from the right at rational points; i.e. for every pair of rational numbers r and s, there is an $\epsilon(r, s) > 0$ so that $R(w, \cdot, \cdot)$ is constant on $[r, r+\epsilon) \times [s, s+\epsilon)$.*

Conversely, if $R(w, r, s) = p/q$ (where p/q is reduced) and the biggest power of consecutive a's in w (resp. b's) is a^m (resp. b^n), then $R(w, r+1/mq, s) \geq p/q+1/q^2$ (resp. $R(w, r, s+1/nq) \geq p/q + 1/q^2$).

Proof. By Lemma 2.14 and Lemma 3.3, it follows that $R(w, \cdot, s)$ is continuous from the right. If s is rational, it takes only finitely many values, by Theorem 3.2. Hence it is locally constant from the right. Hence for any r there is rational $r' > r$ with $R(w, r', s) = R(w, r, s)$. Similarly, there is a rational $s' > s$ with $R(w, r', s') = R(w, r', s)$. Monotonicity (i.e. Lemma 3.3) proves the existence of an ϵ as in the statement of the theorem.

Conversely, let W be admissible with $R(w, r, s, W) = p/q$, where $r = u/v$. Each time we read a^m we hop over $mu + 1$ X's. In the course of a periodic orbit, the end of this a-hop lands on q distinct X's in W, which we label $X_1, X_2, \cdots, X_q$. For some i there are at most $\lfloor v/q \rfloor$ X's in the "interval" $(X_i, X_{i+1}]$, by the pigeonhole principle. Let W' be obtained from W by replacing each X by X^{mq}. Then by comparing orbits, $R(w, u/v+1/mq, s) \geq R(w, (umq+v)/vmq, s, W') \geq p/q+1/q^2$, as claimed. $\square$

It follows that for all positive w, there is an open, dense set in the r–s plane where $R(w, r, s)$ is locally constant and takes values in $\mathbb{Q}$. This is a new manifestation of the familiar phenomenon of phase locking; it would be very interesting to investigate, for a fixed w, how the maximal ϵ as in Theorem 3.7 depends on r and s. For fixed w, a natural guess (in view of the inequality in the second half of Theorem 3.7) is $\epsilon \sim q^{-1}$, where p/q is the locally constant value of $R(w, \cdot, \cdot)$. There is some experimental evidence for this, but it seems hard to prove rigorously.

Example 3.8. The inequality in Theorem 3.7 is sharp for every q for $w = a^m b^n$, as follows from Theorem 3.9, to be proved in § 3.3.

3.3. **The case** $w = ab$. In this section we analyze a particular important special case in detail, namely the case $w = ab$. We derive a concrete formula for $R(ab, p_1/q_1, p_2/q_2)$, and thereby give a new (and much simpler) proof of the main conjecture of Jankins–Neumann [12], first proved by Naimi [17].

Theorem 3.9 (*ab* Theorem). *For $0 \leq r, s < 1$ there is an equality*

$$R(ab, r, s) = \sup_{p_1/q \leq r,\, p_2/q \leq s} (p_1 + p_2 + 1)/q$$

Proof. By monotonicity, it suffices to prove the conjecture for rational r, s. So let $0 \leq p_1/q_1 < 1$ and $0 \leq p_2/q_2 < 1$ be arbitrary. Let W be a cyclic XY word admissible for q_1, q_2 with $R(ab, p_1/q_1, p_2/q_2) = R(ab, p_1/q_1, p_2/q_2, W) = n/m$. We assume n/m is reduced, so that ab has an orbit of period m on W. We decompose

W into m subwords $W = T_1 T_2 \cdots T_m$. For each i, let T_i^+ denote the rightmost letter of T_i. Then $ab(T_i^+) = T_{n+i}^+$, indices taken mod m.

There is another (cyclic) decomposition of W into m subwords $U_1 U_2 \cdots U_m$ such that $b(T_i^+) = U_i^+$ and $a(U_i^+) = T_{n+i}^+$, where we denote the rightmost letter of U_i by U_i^+. Each U_i^+ is therefore a Y and each T_i^+ is an X, so the set of endpoints of these words are *disjoint*. Let $V_1 V_2 \cdots V_{2m}$ be the common refinement.

We now show that we can adjust the letters in W without affecting the dynamics. First of all, any reordering of the letters in each V_i which leaves the last letter intact will not affect the dynamics, so without loss of generality we assume that V_i is of the form $X^x Y^y$ if $V_i^+ = U_j^+$ for some j (possibly with $x = 0$), and $V_i = Y^y X^x$ if $V_i^+ = T_j^+$ for some j (possibly with $y = 0$).

Next, suppose some T_i is entirely contained in some U_j, so that $T_i = V_k$ for some k, and $T_i = Y^y X^x$. Notice V_{k-1} is also entirely contained in U_j, and $V_{k-1}^+ = T_{i-1}^+$. We claim that moving the (possibly empty) string Y^y left to the start of V_{k-1} will not decrease the rotation number of W. First, we still have $a(U_l^+) = T_{n+l}^+$ for all l (since a ignores Y's). Second, we have $b(T_l^+) \geq U_l^+$ for all l, since the number of Y's between T_l^+ and U_l^+ is either the same or is decreased by this transformation. This proves the claim.

It follows that whenever we have a string of consecutive T_i's entirely contained in some U_j, we can move the Y's out of the T_i's and to the left side of U_j. After finitely many transformations of this kind, we can assume that every U_j is of the form $Y^{z_j} X^{x_j} Y^{y_j}$ (where *a priori* possibly x_j and/or z_j are 0).

Now, we claim that in fact every $x_j > 0$. For, otherwise, there is some U_j which consists entirely of Y's. But then $T_{n+j-1}^+ = a(U_{j-1}^+) = a(U_j^+) = T_{n+j}^+$ which is absurd. The claim follows. In particular, we can conclude that there is some l so that $U_{i+l}^+ < T_{n+i}^+ < U_{i+l+1}^+$ for all i.

But from this the theorem follows easily. Since $a(U_i^+) = T_{n+i}^+$, from the definition of the a transformation we get an inequality $p_1 + 1 \geq x_i + x_{i+1} + \cdots + x_{i+l-1} + 1$ and therefore $p_1 \geq \sum_{j=0}^{l-1} x_{i+j}$. Since this is true for every i, and since $\sum_{j=1}^{m} x_j = q_1$ we conclude $p_1/q_1 \geq l/m$. Similarly, $p_2/q_2 \geq (n - l - 1)/m$. But $R(ab, l/m, (n - l - 1)/m, (XY)^m) = n/m$ and the theorem is proved. $\square$

The proof of Theorem 3.9 is expressed in terms of the purely combinatorial question of maximizing $R(ab, p_1/q_1, p_2/q_2, W)$ over admissible q_1, q_2 words W. It turns out that for the case of ab there is a simple formula relating R to r which solves the combinatorial question of minimizing $R(ab, p_1/q_1, p_2/q_2, W)$ over admissible q_1, q_2 words W. This is the following duality formula:

Proposition 3.10 (Duality formula). *Suppose p_1/q_1 and p_2/q_2 are reduced fractions. There is a formula*

$$r(ab, p_1/q_1, p_2/q_2) + R(ab, (q_1 - p_1 - 1)/q_1, (q_2 - p_2 - 1)/q_2) = 2$$

Proof. Without loss of generality we can assume $0 \leq p_i/q_i < 1$. We have $R(ab, (p_1 - q_1)/q_1, (p_2 - q_2)/q_2) = R(ab, p_1/q_1, p_2/q_2) - 2$. On the other hand, $R(ab, (p_1 - q_1)/q_1, (p_2 - q_2)/q_2)$ may be calculated as the maximum, over all W admissible for q_1, q_2 of n/m, where we alternate between moving *left* $q_1 - p_1$ X's and $q_2 - p_2$ Y's. Changing the orientation of the circle, this evidently computes $-r(ab, (q_1 - p_1 - 1)/q_1, (q_2 - p_2 - 1)/q_2)$. $\square$

Figure 1 shows the graph of $R(ab, \cdot, \cdot)$ over $[0,1] \times [0,1]$. Discontinuities of R are represented by vertical walls. Because of the monotonicity of R, a camera situated at $(-1, -1, 3)$ can see the entire graph without occlusion. Because of its stepwise nature, we refer to such graphs as *ziggurats*.
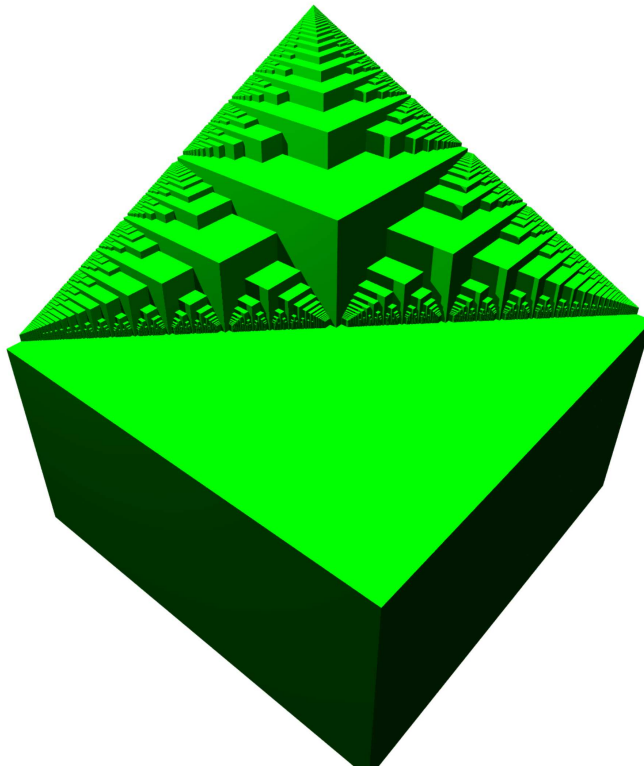

FIGURE 1. The Jankins–Neumann ziggurat (i.e. the graph of $R(ab, \cdot, \cdot)$).

3.4. **Stairsteps.** We have described an algorithm to compute R. However, this algorithm as stated is very inefficient because of the very large number of admissible q_1, q_2 words for large q_i. Given a positive word w and a fixed p_1/q_1, the function $t \to R(w, p_1/q_1, t)$ only takes on finitely many values in the interval $t \in [0, 1)$ (for, it is *a priori* bounded, and rational with denominator $\leq q_1$). Moreover, we already know that this function is nondecreasing as a function of t, and therefore it is completely specified if we know the finitely many values that are achieved, and the infimal t_i at which each value is achieved.

The following theorem says that these finitely many values are *rational*, that each value is achieved at some *minimal* t_i, and gives an algorithm to compute them.

Theorem 3.11 (Stairstep Theorem). *Let w be positive, and suppose we are given rational numbers p/q and c/d so that c/d is a value of $R(w, p/q, \cdot)$ (so necessarily $d \leq q$). Then $\inf\{t : R(w, p/q, t) = c/d\}$ is rational, and there is an algorithm to compute it. Moreover, if u/v is this infimal value, $R(w, p/q, u/v) = c/d$.*

Proof. Given w, p/q and c/d we compute the infimal t such that $R(w, p/q, t) \geq c/d$; this will give us the same answer. It suffices to compute the infimum over rational $t = u/v$; we give an algorithm to do this whose output is evidently rational. Without knowing u and v in advance, we let W be an admissible q, v word for which $R(w, p/q, u/v) \geq c/d$ is achieved. We write W as $XY^{t_1} XY^{t_2} \cdots XY^{t_q}$ where the t_i are (for the moment) real variables subject to linear constraints $t_i \geq 0$ and $\sum t_i = v$.

We assume c/d is reduced. Suppose equality is achieved, so that w has a periodic orbit of period d. After replacing w by a cyclic permutation we assume it begins with a string of b's and ends with a string of a's, so that the periodic orbit begins in the terminal string of Y's. We measure the number of X's we jump over or land on at each step. With each maximal a^m string we jump over precisely $mp + 1$ X's, landing on the last one. With each maximal b^m string, if we start at the ith X, we jump over l X's where l is the smallest number such that $t_i + t_{i+1} + \cdots + t_{i+l} \geq mu + 1$. We can rewrite this condition by saying that l is the *biggest* number such that $t_i + t_{i+1} + \cdots + t_{i+l-1} \leq mu$; the advantage of this reformulation is that this latter inequality is *homogeneous*. Note that we must allow the possibility $l = 0$ (if $t_i > mu$), in which case this inequality is vacuous. If we only have $R(w, p/q, u/v) \geq c/d$ then this inequality still holds, but we do not assume l is the biggest number with this property.

Write $w^d = b^{\beta_k} a^{\alpha_k} \cdots b^{\beta_1} a^{\alpha_1}$ (if w is primitive, the indices are periodic with period k/d). Then we will apply k strings of a's and b's. Let l_i for $1 \leq i \leq k$ be the value of l as above when we apply the string b^{β_i}. Then we obtain an equality $\sum_{i=1}^{k}(\alpha_i p + 1 + l_i) = cq$. For each i define $s_i = \sum_{j=1}^{i}(\alpha_j p + 1) + \sum_{j=1}^{i-1} l_j$. Then our inequalities are precisely of the form $t_{s_i} + t_{s_i+1} + \cdots + t_{s_i+l_i-1} \leq \beta_i u$, indices taken mod q.

Since our system of inequalities is homogeneous, linear and defined over $\mathbb{Z}$, we can find a solution in integers iff we can find a solution over the reals. We rescale so that $v = 1$. Our algorithm then has the following form: First, enumerate all non-negative integral solutions to $\sum_{i=1}^{k} l_i = cq - \sum_{i=1}^{k}(\alpha_i p + 1)$ (there are only finitely many such solutions, equal to the number of ordered partitions of $cq - \sum_{i=1}^{k}(\alpha_i p + 1)$ into k non-negative integers). For each such solution, define s_i by the formula $s_i = \sum_{j=1}^{i}(\alpha_j p + 1) + \sum_{j=1}^{i-1} l_j$. Then let u be the smallest real number (necessarily rational) such that the system of equations $\{\sum_{i=1}^{q} t_i = 1$ and for each $1 \leq i \leq k$, $t_i \geq 0$ and $t_{s_i} + t_{s_i+1} + \cdots + t_{s_i+l_i-1} \leq \beta_i u\}$ has a solution (indices taken mod q). The smallest u over all such collections of l_i is the desired quantity. Evidently it is rational, and achieves the smallest possible value of $R(w, p/q, \cdot)$ which is $\geq c/d$. If c/d is achieved, $R(w, p/q, u) = c/d$. $\square$

An interesting special case of the stairstep theorem is when $d = q$.

Proposition 3.12. *Suppose there is some t for which $R(w, p/q, t) = c/q$, where p, c are coprime to q. Let $w = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_m} b^{\beta_m}$. Then $R(w, p/q, t) = c/q$ for t on some interval $[u/nq, u/nq + \epsilon)$ where u/n is the least rational for which*

$$c - m - h_a(w)p = \sum_{i=1}^{m} \left\lfloor \frac{\beta_i u + \epsilon}{n} \right\rfloor$$

Proof. Let $W = XY^{t_1} XY^{t_2} \cdots XY^{t_q}$ as in the proof of Theorem 3.11. Since w has a periodic orbit of period exactly q, any b-string starting on adjacent X's must land in adjacent Y^* strings. This dramatically cuts down on the number of partitions

we need to consider; for instance, the l_i as in Theorem 3.11 are periodic with period k/d.

Because of this, the constraints for the linear programming problem are invariant under cyclic permutation of the variables t_i, and by convexity, setting all t_i equal gives an extremal solution. The result follows   □

Proposition 3.12 gives rise to the following inequality:

Proposition 3.13. *For any positive word w of the form $w = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_m} b^{\beta_m}$, if $R(w, p/q, t) = c/q$ there is an inequality*

$$|c/q - h_a(w)p/q - h_b(w)t| \leq 2m/q$$

Proof. We always have $R(w, p/q, t) \geq h_a(w)p/q + h_b(w)t$ coming from the representation with $a = R_{p/q}$ and $b = R_t$, so we just need to prove the inequality for $t = u/nq$ as in Proposition 3.12.

We have

$$c/q = m/q + h_a(w)p/q + \frac{1}{q} \sum_{i=1}^{m} \left\lfloor \frac{\beta_i u + \epsilon}{n} \right\rfloor$$

Since $\sum \beta_i u / qn = h_b(w)u/nq$ the inequality follows.   □

If q is big compared to m, Proposition 3.13 says that $R(w, p/q, t)$ is very close to $h_a(w)p/q + h_b(w)t$, which is achieved for the linear representation $a = R_{p/q}$, $b = R_t$. Combined with the inequality in Theorem 3.7, we obtain strong constraints on the values of $R(w, p/q, \cdot)$ achieved, especially if w contains a substring of the form a^m with m large.

It is natural to wonder whether the inequality in Proposition 3.13 can be generalized to cases when the denominator of $R(w, r, s)$ is strictly smaller than the denominators of r and s; we return to this question in § 3.7.

3.5. **Speeding up the computation.** The algorithm described in the proof of Theorem 3.11 has one big bottleneck, namely the need to enumerate the partitions of $cq - \sum_{i=1}^{k}(\alpha_i p + 1)$ into k non-negative integers l_i. This number of partitions is exponential in k, which is itself linear in w and $d \leq q$. So the runtime of the algorithm above is *a priori* exponential in w and q.

Apart from the inequalities $\sum_{i=1}^{q} t_i = 1$ and $t_i \geq 0$, we are left with the "s-inequalities" which are all of the form $t_{s_i} + t_{s_i+1} + \cdots + t_{s_i+l_i-1} \leq \beta_i u$. For each i with $1 \leq i \leq k$ let J_i denote the "interval" $[s_i, s_i + l_i - 1]$ in the "circle" $\mathbb{Z}/q\mathbb{Z}$. A partition $L := l_1, l_2, \cdots, l_k$ of $cq - \sum_{i=1}^{k}(\alpha_i p + 1)$ thereby determines a combinatorial configuration of intervals in a circle (weighted by integers β_i), and the minimal $u(L)$ depends only on the combinatorics of this (weighted) configuration.

In fact, we consider vectors of non-negative reals r_i so that the weighted sum of indicator functions $\chi(r) := \sum r_i \chi_{J_i}$ is ≥ 1 everywhere in the circle. The s-inequalities imply $u \geq 1/(\sum r_i \beta_i)$, and linear programming duality gives the equality $u(L) = \sup_{\sum r_i \chi_{J_i} \geq 1} 1/(\sum r_i \beta_i)$. Hence the problem becomes to minimize $\sum r_i \beta_i$ subject to $\sum r_i \chi_{J_i} \geq 1$; informally, to cover S^1 as "efficiently as possible" with the intervals J_i. If we call the minimum of $\sum r_i \beta_i$ the *efficiency* of a covering, then since $u = \min_L u(L)$, we seek the partition giving rise to the *covering of least efficiency*.

A *partial* partition is a vector $K := l_1, l_2, \cdots, l_j$ with $j < k$ and $\sum_{i=1}^{j} l_i < cq - \sum_{i=1}^{k}(\alpha_i p + 1)$; such a partial partition can be extended to a (complete) partition

L as above in potentially many ways; we write $K < L$ if L extends K to a complete partition. For a partial partition K, define $u(K) = \sup 1/(\sum r_i \beta_i)$ over all coverings of S^1 by intervals J_i with $1 \leq i \leq j$; equivalently, we let $u(K)$ be the smallest u for which there is a feasible solution to the linear programming problem determined by the s-inequalities coming from K. Each additional s-inequality can only reduce the space of feasible solutions, and therefore $u(K) \leq u(L)$ for every extension $K < L$. But $u = \min_L u(L)$. Consequently, if there is a complete partition L' with $u(K) > u(L')$, then $u(L) > u(L')$ for every $K < L$, and therefore we can ignore all extensions L of K when computing u.

Example 3.14. Since we assume *a priori* that $u \leq 1$, we must have $l_i \leq q\beta_i$ for all i.

If we enumerate partitions lexicographically, inequalities as above let us prune the tree of partitions and speed up the computation of u. Figure 2 shows the ziggurat for $w = abaab$, which is computed using a combination of methods.
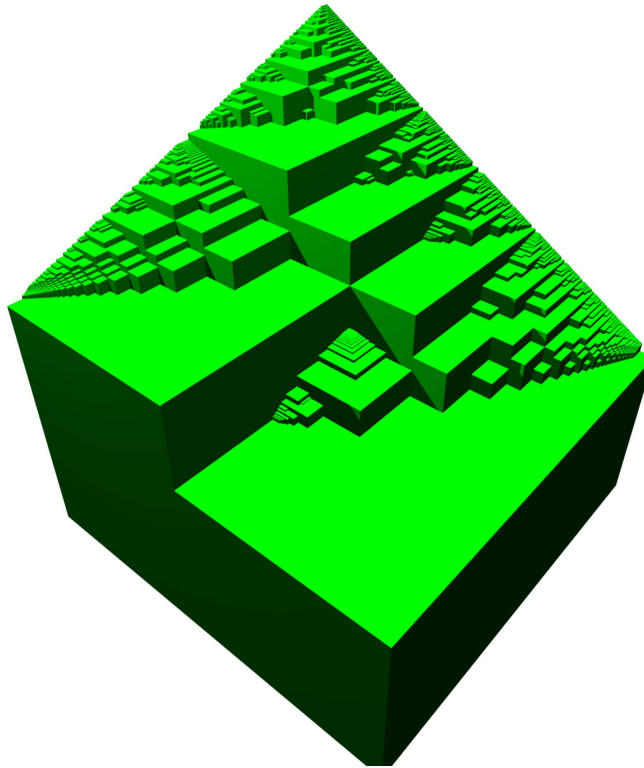

FIGURE 2. The *abaab* ziggurat.

3.6. **Limits and rotations.** We introduce the notation $R(w, p_1/q_1, p_2/q_2-)$ to mean the limit of $R(w, p_1/q_1, t)$ as $t \to p_2/q_2$ from below. Since $R(w, p_1/q_1, t)$ is monotone nondecreasing as a function of t, and takes only rational values with

denominators bounded by q_1, this limit is well-defined and rational, with denominator bounded by q_1, and is achieved on a semi-open interval of values $[u/v, p_2/q_2)$ where u/v can be determined by Theorem 3.11. We similarly introduce the notation $R(w, p_1/q_1-, p_2/q_2)$ and $R(w, p_1/q_1-, p_2/q_2-)$.

Proposition 3.15. *Let w be positive, and suppose we are given rational numbers p_1/q_1 and p_2/q_2. Then $R(w, p_1/q_1-, p_2/q_2)$ is the supremum of $\mathrm{rot}^\sim(w)$ for all representations with a conjugate to the rotation R_{p_1/q_1} and $\mathrm{rot}^\sim(b) = p_2/q_2$ (and similarly with the roles of a and b interchanged), and $R(w, p_1/q_1-, p_2/q_2-)$ is the supremum of $\mathrm{rot}^\sim(w)$ for all representations with a conjugate to R_{p_1/q_1} and b conjugate to R_{p_2/q_2}.*

Proof. We first claim that for any a with $\mathrm{rot}^\sim(a) < p_1/q_1$ there is some α conjugate to R_{p_1/q_1} satisfying $\alpha(p) \geq a(p)$ for all p. As in the proof of Lemma 2.17 we can produce a smooth family a_t interpolating between a^- and a^+, all C^0 close to a. Some a_t with $a_t(p) \geq a(p)$ has irrational rotation number, and is therefore conjugate (by some g) to some R_θ with $\mathrm{rot}^\sim(a) \leq \theta < p_1/q_1$. If we let α be obtained by conjugating R_{p_1/q_1} by g then α satisfies the desired properties.

Conversely, any homeomorphism conjugate to R_{p_1/q_1} can be C^0 approximated by homeomorphisms conjugate to R_t for any $t < p_1/q_1$. $\square$

An important special case is $1-$. Because R_1 is central, for any positive w we have $R(w, 1-, p/q) = h_a(w) + h_b(w)p/q$ and $R(w, p/q, 1-) = h_a(w)p/q + h_b(w)$. This gives rise to straight lines of slope 1, clearly visible in the graphs of $R(ab, \cdot, \cdot)$ and $R(abaab, \cdot, \cdot)$ in Figures 1 and 2.

The Stairstep Theorem (i.e. Theorem 3.11) implies that for each w and each p/q there is some smallest u/v with $R(w, p/q, u/v) = h_a(w)p/q + h_b(w)$ (and similarly with the roles of a and b interchanged). Proposition 3.12 says that $u/v = u/nq$ where u/n is minimized subject to

$$h_b(w)q - m = \sum_{i=1}^{m} \left\lfloor \frac{\beta_i u + \epsilon}{n} \right\rfloor$$

Note that the result does not depend on p; this "explains" why the fringes of the ziggurats appear periodic on every scale near the sloped edges. Solving for integers u and n to minimize u/n is straightforward.

Example 3.16. For $w = ab$, $R(ab, p/q, t) = 1 + p/q$ on the maximal interval $t = [(q-1)/q, 1)$ (this follows from Theorem 3.9, of course).

Example 3.17. For $w = abaab$, $R(abaab, p/q, t) = 2 + 3p/q$ for 3 coprime to q on the maximal interval $t = [(q-1)/q, 1)$, and $R(abaab, t, p/q) = 3 + 2p/q$ for odd q on the maximal interval $t = [(2q-1)/2q, 1)$.

Without the condition that $h_a(w)$ and q are coprime, the formula is more tricky; for example, $R(abaab, 2/3, t) = 4$ on the maximal interval $t = [1/2, 1)$, and $R(abaab, 5/6, t) = 9/2$ on the maximal interval $t = [3/4, 1)$.

Example 3.18. We have $R(abaab, 1/2-, t) = 1 + R(abb, 1/2-, t) = 1 + R(ab, 1/2-, 2t)$ which is equal to $2 + 1/(2p+1)$ for $2t \in [(p+1)/(2p+1), p/(2p-1))$ for $p > 1$, and $2 + p/(2p+1)$ for $2t \in [2p/(2p+1), (2p+2)/(2p+3))$.

On the other hand, $R(abaab, t, 1/2-) = 1 + R(abaab^{-1}, t, 1/2-)$ where the notation here simply means representations for which b is conjugate to $R_{1/2}$. Obviously $R(abaab^{-1}, t, 1/2-) \leq R(ab, t, 2t)$. On the other hand, by the proof of

Theorem 3.9, we know that for $t = p/q$ with q odd, the extremal q, q word for $R(ab, p/q, 2p/q)$ is $(XY)^q$; evidently the dynamics of a^2 and b are conjugate in this case by an element which is itself conjugate to $R_{1/2}$, so we obtain an equality $R(abaab, p/q, 1/2-) = 1 + R(ab, p/q, 2p/q)$ for q odd (and in fact by the same reasoning, $R(aba^n b, p/q, 1/2-) = 1 + R(ab, p/q, cp/q)$ for q odd and n coprime to q).

3.7. Slippery points.

Definition 3.19. Let w be positive. A pair of rational numbers (r, s) is *slippery* for w if there is a *strict* inequality $R(w, r', s') < R(w, r-, s-)$ for all $r' < r$, $s' < s$.

If (r, s) is not slippery, then there are $r' < r$ and $s' < s$ with $R(w, r', s') = R(w, r-, s-)$. It follows that this common value is achieved on the entire region $[r', r) \times [s', s)$ and is therefore rational.

Example 3.20. $(1, t)$ and $(t, 1)$ are slippery for all positive words. $(1/2, 1/2)$ is slippery for *abaab*, by Example 3.18. Experimentally, $(1/2, 1/2)$ is slippery for many other words; e.g. *abaababb*, *abaabaaaabb*.

The following conjecture generalizes Proposition 3.13, and is supported by some experimental evidence.

Slippery Conjecture. *For any positive w of the form $w = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_m} b^{\beta_m}$, if $R(w, r, s) = p/q$ where p/q is reduced, then $|p/q - h_a(w)r - h_b(w)s| \le m/q$.*

Note that for the "linear" representation in which $a = R_r$ and $b = R_s$, we have $\mathrm{rot}^\sim(w) = h_a(w)r + h_b(w)s$. So this conjecture says that the bigger the denominator of $R(w, r, s)$, the closer the extremal representation (one realizing $\mathrm{rot}^\sim(w)$) must be to the linear representation. The idea behind this conjecture is that the "more nonlinear" an extremal representation is, the more rigid it is, and the smaller the denominator of $R(w, r, s)$. Contrapositively, the bigger the denominator, the less rigid (and hence the more slippery) and the closer to linear. This is only heuristic reasoning (and hand-wavy at that), but it is in sympathy with the Stability Theorem.

This conjecture is very interesting in view of the following consequence:

Proposition 3.21. *The Slippery Conjecture implies that for any slippery (r, s), we have $R(w, r-, s-) = h_a(w)r + h_b(w)s$. In particular, it implies that $R(w, r-, s-)$ is rational for all rational (r, s).*

Proof. If (r, s) is slippery, there are $r' < r$, $s' < s$ arbitrarily close to r, s for which the denominator of $R(w, r', s')$ is arbitrarily large. Since $R(w, r-, s-)$ is rational for (r, s) rational and not slippery, the proposition follows. $\square$

The Slippery Conjecture has been experimentally checked for several words, up through the range $q \le 14$, which is close to the limit of what can be computed easily. Figures 3 and 4 show the range of experimentally computed values of $|p/q - h_a(w)r - h_b(w)s|$ for words *abaab* and *abaabbabbbababaab* respectively, for $q \le 14$. In each case, the maximal error m/q is precisely achieved for "most" q.

Note that for any w we have $R(w, 1/2-, 1/2-) = (h_a(w) + h_b(w))/2$. For, any representation in which both a and b are conjugate to rigid rotations with rotation number $1/2$ factors through the infinite dihedral group, which is amenable. On any representation which factors through an amenable group, rotation number becomes
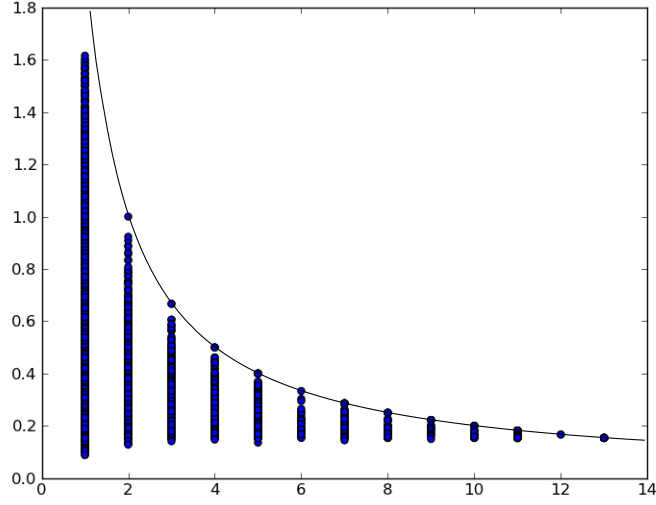
FIGURE 3. Scatter plot of $|R(w, r, s) - h_a(w)r - h_b(w)s|$ versus q
for $w = abaab$ for all rational values of r and s with denominator
at most 13. The graph $m/q = 2/q$ is also pictured.

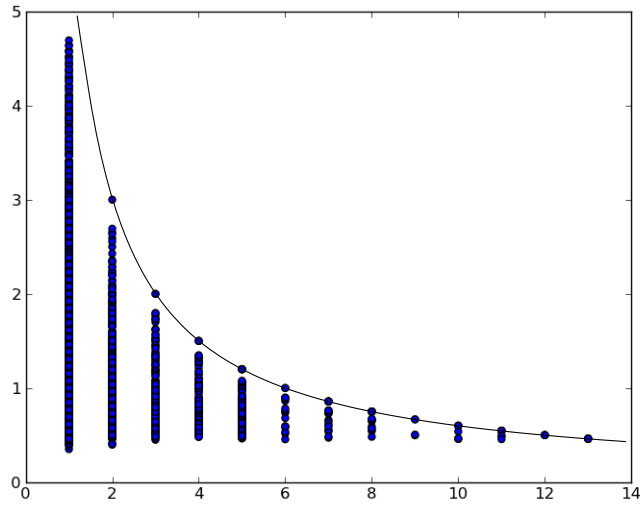a *homomorphism*. This "explains" why $(1/2, 1/2)$ is likely to be slippery for many
w.

FIGURE 4. Scatter plot of $|R(w, r, s) - h_a(w)r - h_b(w)s|$ versus q
and $m/q = 6/q$ for $w = abaabbabbbababaab$.

3.8. **Immersions and** scl**.** We now describe an unexpected connection to hyperbolic geometry and stable commutator length. For the benefit of the reader, we quickly recall some basic elements of the theory of stable commutator length (for a more substantial introduction see [5]).

Definition 3.22. If G is a group, and $g \in [G, G]$, the commutator length $\mathrm{cl}(g)$ is the least number of commutators in G whose product is g, and the *stable commutator length* is the limit $\mathrm{scl}(g) := \lim_{n \to \infty} \mathrm{cl}(g^n)/n$. If $g_1, g_2, \cdots, g_m$ is a collection of elements with $\prod_i g_i \in [G, G]$ then $\mathrm{scl}\left(\sum g_i\right) := \lim_{n \to \infty} \mathrm{cl}\left(\prod_i g_i^n\right)/n$.

Definition 3.23. If G is a group, a *homogeneous quasimorphism* is a function $\phi : G \to \mathbb{R}$ satisfying $\phi(g^n) = n\phi(g)$ for all $g \in G$ and $n \in \mathbb{Z}$, and for which there is a least non-negative real number $D(\phi)$ (called the *defect*) such that $|\phi(gh) - \phi(g) - \phi(h)| \le D(\phi)$ for all $g, h \in G$.

Generalized Bavard Duality ([5], Thm. 2.79) says that for any G, and for any finite set of elements g_i with $\prod g_i \in [G, G]$, there is an equality

$$\mathrm{scl}\left(\sum_i g_i\right) = \sup_\phi \left(\sum \phi(g_i)\right)/2D(\phi)$$

where the supremum is taken over all homogeneous quasimorphisms.

Any representation $\rho : G \to \mathrm{Homeo}^+(S^1)$ gives rise to a homogeneous quasimorphism on G, namely rotation number. In general, this homogeneous quasimorphism satisfies $D(\phi) \le 1$, where generically $D(\phi) = 1$. Therefore stable commutator length can be used to give an upper bound on R. However what is quite surprising is that this upper bound is often *sharp*, under geometrically meaningful conditions. This is the content of the next theorem.

Theorem 3.24. *Let γ_w be the unique geodesic representative of the word w in the hyperbolic (q_1, q_2, ∞)-orbifold $O(q_1, q_2)$. If γ_w virtually bounds an positively immersed subsurface S in $O(q_1, q_2)$ then*

$$R(w, 1/q_1-, 1/q_2-) = h_a(w)/q_1 + h_b(w)/q_2 + \mathrm{area}(S)/2\pi$$

In general there are inequalities

$$R(w, 1/q_1-, 1/q_2-) \ge h_a(w)/q_1 + h_b(w)/q_2 + A(\gamma_w)/2\pi$$

where $A(\gamma_w)$ is the algebraic area enclosed by γ_w, and

$$R(w, 1/q_1-, 1/q_2-) \le h_a(w)/q_1 + h_b(w)/q_2 + 2\,\mathrm{scl}_{G(q_1,q_2)}(w)$$

where scl *(denoting stable commutator length) is computed in the group $G(q_1, q_2) := \langle a, b \mid a^{q_1} = b^{q_2} = 1 \rangle$.*

Proof. Let $H(q_1, q_2)$ be the central extension of $G(q_1, q_2)$, defined by the presentation $H(q_1, q_2) := \langle a, b, t \mid [t, a] = [t, b] = 1, a^{q_1} = t, b^{q_2} = t \rangle$. $G(q_1, q_2)$ is the (orbifold) fundamental group of the (q_1, q_2, ∞)-orbifold $O(q_1, q_2)$, and $H(q_1, q_2)$ is the fundamental group of its (orbifold) unit tangent bundle.

The (unique complete finite area) hyperbolic structure on $O(q_1, q_2)$ gives rise to a representation $G \to \mathrm{PSL}(2, \mathbb{R})$ which can be thought of as a subgroup of $\mathrm{Homeo}^+(S^1)$. This is covered by a representation $H \to \mathrm{Homeo}^+(S^1)^\sim$. The elements a and b are conjugate to rigid rotations, and for this representation $\mathrm{rot}^\sim(w) - h_a(w)/q_1 - h_b(w)/q_2$ is equal to $A(\gamma_w)/2\pi$ (this essentially follows from the Gauss-Bonnet theorem). This gives one inequality.

Generalized Bavard duality (applied to any representation) and the fact that
the defect of any rotation quasimorphism is at most 1, gives the upper bound
$R(w, 1/q_1-, 1/q_2-) - h_a(w)/q_1 - h_b(w)/q_2 \le 2 \operatorname{scl}_H(w - h_a(w)a - h_b(w)a)$. If
γ_w virtually bounds a positively immersed surface S in O, this surface and the
rotation quasimorphism are both extremal for w (see [4] for details). In this case,
both inequalities are equalities (with $\operatorname{area}(S) = A(\gamma_w)$).

Since H is an amenable extension of G, and $H^2(G; \mathbb{R}) = 0$, the projection from H
to G is an isometry for scl, so $\operatorname{scl}_H(w - h_a(w)a - h_b(w)b) = \operatorname{scl}_G(w - h_a(w)a - h_b(w)b)$.
But in G, the elements a and b have finite order, so this last term is equal to
$\operatorname{scl}_G(w)$.   □

Remark 3.25. The group $G(q_1, q_2)$ is virtually free, and therefore scl is rational and
can be computed in polynomial time (for fixed q_1, q_2) by the method of [3] (also
see [5], Chapter 4). In fact, an algorithm due to Walker lets one compute scl in
polynomial time in word length *and in q_1, q_2*.

Moreover, for fixed q_1, q_2, the function $w \to A(\gamma_w)$ is an example of a *bicombable
function* (see [6]), and can be computed in *linear* time. This makes it practical to
actually compute the bounds in Theorem 3.24.

Remark 3.26. The question of which geodesics virtually bound immersed subsur-
faces of hyperbolic surfaces is very subtle and difficult. This question is pursued in
some detail in [4], and in the special case of the modular orbifold in [7].

Example 3.27. The special case $w = ab$ is particularly simple. In this case, γ_{ab}
does not have a geodesic representative and one must work with the cusp repre-
sentative instead, which is the boundary of the (q_1, q_2, ∞) orbifold, and therefore
always tautologically bounds $O(q_1, q_2)$. Hence $R(ab, 1/q_1-, 1/q_2-) = 1/q_1 + 1/q_2 +
\operatorname{area}(O(q_1, q_2))/2\pi = 1$.

In order to apply Theorem 3.24 for general $R(w, p_1/q_1-, p_2/q_2-)$ we use the
following fact. Let $\phi_{p_1, p_2} : F \to F$ be defined on generators by $a \to a^{p_1}$ and
$b \to b^{p_2}$. Then $R(\phi_{p_1, p_2}(w), 1/q_1, 1/q_2) = R(w, p_1/q_1, p_2/q_2)$ and similarly with
p_i/q_i replaced by p_i/q_i-.

Example 3.28 ($a^{p_1} b^{p_2}$). $R(ab, p_1/q_1-, p_2/q_2-) = R(a^{p_1} b^{p_2}, 1/q_1-, 1/q_2-)$. Let $G =
G(q_1, q_2)$ as above. Now, $\operatorname{scl}_G(a^{p_1} b^{p_2}) = \operatorname{scl}_G(ab) = 1/2 - 1/2q_1 - 1/2q_2$ so we get
the inequality $R(ab, p_1/q_1-, p_2/q_2-) \le 1 + (p_1 - 1)/q_1 + (p_2 - 1)/q_2$. This is never
sharp if at least one $p_i > 1$, and therefore *none* of the geodesics $\gamma_{a^{p_1} b^{p_2}}$ virtually
bound a positively immersed surface in $O(q_1, q_2)$ except for γ_{ab}.

3.9. **Isobars.** In this section we prove a structure theorem for the level sets of R.
But first we must prove a couple of lemmas.

Lemma 3.29. *Let $I_1, I_2, \cdots, I_k$ be a finite set of closed intervals in S^1, and for
each i let I_i^- be the initial point and I_i^+ the final point of the interval (with respect
to the orientation on S^1). The set of values of s for which there is a homeomor-
phism $a(\cdot)$ conjugate to a rotation R_s, and such that $a(I_i^-) = I_i^+$ for all i, is a
connected interval (possibly open or half-open) with rational endpoints. Moreover,
the endpoints depend only on the combinatorics of the I_j, and can be computed by
linear programming.*

Proof. A rotation number s achieved by some a as above is said to be *feasible* for
the collection of intervals. It is obvious from the definition that the set of feasible
s for a given collection of intervals depends only on the combinatorics.

The I_j and their complements partition S^1 into disjoint intervals J_i. We assign a variable t_i to each J_i. Showing s is feasible is equivalent to the feasibility of the linear programming problem defined by the following constraints:

  (1) each t_i is in $(0,1)$
  (2) for each I_j the set of t_i for which the $J_i \subset I_j$ sums to s
  (3) the sum of all the t_i is 1

The connectedness of the set of feasible s follows from convexity. Rationality follows from the form of the linear programming problem. The proof follows. $\qquad\square$

Remark 3.30. The set of s feasible for a given finite collection of intervals could easily be empty; for example, if one interval is properly contained in another.

Recalling the definition of $X(w, r, s)$ as in Lemma 2.16, we introduce the notation $X(w : t)$ for the set of r, s for which $t \in X(w, r, s)$. Evidently $X(w : t)$ is the intersection of the set of r, s with $R(w, r, s) \geq t$ with the set of r, s with $R(w, -r, -s) \geq -t$.

Lemma 3.31. *Let w be positive. $X(w : t)$ is the closure of the set of pairs (r, s) such that there is a representation with a conjugate to a rotation R_r and b conjugate to a rotation R_s, and with $\mathrm{rot}^\sim(w) = t$.*

Proof. Suppose $(r, s) \in X(w : t)$. As u varies in $(-\epsilon, \epsilon)$, the maximum of $X(w, r + u, s + u)$ is continuous from the right, and the minimum is continuous from the left. In order to complete the proof we make two observations.

On the one hand, by Lemma 2.18, for (r, s) irrational Herman numbers, any t in the interior of $X(w, r, s)$ can be achieved by a smooth representation for which a is conjugate to R_r and b is conjugate to R_s.

On the other hand, taking $a = R_r$ and $b = R_s$ on the nose, we see that $h_a(w)r + h_b(w)s \in X(w, r, s)$ for all (r, s).

So we have two possibilities: either there is u arbitrarily close to 0 with $(r + u, s + u)$ both Herman irrationals and t in the interior of $X(w, r + u, s + u)$, or else $h_a(w)r + h_b(w)s = t$; in either case we are done by one of the two observations above. $\qquad\square$

We are now ready to prove the main result of this section.

Theorem 3.32 (Isobar Theorem). *Let w be positive. For any rational p/q the set of $r, s \subset [0,1] \times [0,1]$ such that $R(w, r, s) \geq p/q$ is a finite sided rational polyhedron, whose boundary consists of finitely many horizontal or vertical segments.*

We call the frontier of $R(w, r, s) \geq p/q$ the *p/q isobar*. This is part of the frontier of the level set $R(w, r, s) = p/q$. Note that it is *not* true in general that the closure of the level set itself is a finite sided polyhedron, even for $w = ab$. Thus the set of r, s such that $R(w, r, s) \leq p/q$ can be in general quite complicated. For example, the level set $R(ab, \cdot, \cdot) = 1$ has the line $r + s = 1$ in its frontier, whereas the level set $R(abaab, \cdot, \cdot) = 2$ has infinitely many segments in its frontier. See Figures 1 and 2.

Proof. By the discussion above, it is sufficient to prove that $X(w : p/q)$ is a finite sided rational polygon whose boundary consists of horizontal and vertical segments. By Lemma 3.31 we need to compute the set of r, s for which there are representations with a and b conjugate to rotations through r and s respectively, and $\mathrm{rot}^\sim(w) = p/q$. So consider such a representation.

Let $w = a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_m} b^{\beta_m}$ and let $M = \sum \alpha_i + \sum \beta_i$ be the word length of w. By the hypothesis on the rotation number, we can tile $[0, p]$ by qM intervals, each of which is of the form $[t, a(t)]$ or $[t, b(t)]$. Consider the projection of these qM intervals to the circle; there are a large but finite number of combinatorial types for the image; call such a combinatorial type a *partition*. For each partition, we compute the set of r feasible for the a-intervals and the set of s feasible for the b-intervals, by Lemma 3.29; the set of (r, s) compatible with both is therefore a rectangle with rational vertices. The union of the interiors of these rectangles over all partitions is dense in $X(w : p/q)$; the theorem follows. $\square$

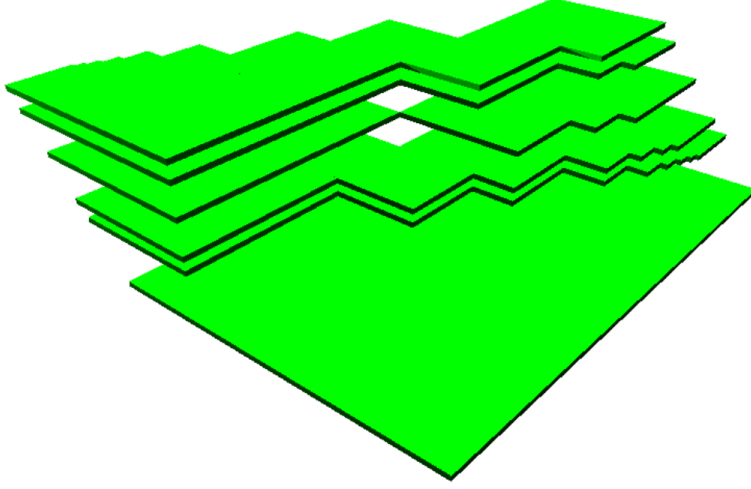Figure 5 depicts $X(abaab : p/q)$ for a few simple values of p/q.

FIGURE 5. Slices $X(abaab : p/q)$ for $p/q = 2, 9/4, 7/3, 5/2, 8/3,$ $11/4$. The interior of the $5/2$ slice is disconnected at a slippery point.

4. ARBITRARY WORDS

4.1. Semipositive words.

Definition 4.1. A word w is *semipositive* if it either contains no a^{-1} or no b^{-1}.

Theorem 3.2 generalizes in a straightforward way to semipositive words.

Theorem 4.2. *Let w be semipositive (without loss of generality, suppose it contains no a^{-1}). If r is rational, so is $R(w, r, s)$. Moreover, the denominator of $R(w, r, s)$ is no bigger than the denominator of r.*

Proof. After replacing w by a cyclic conjugate, we can assume it ends with a. Let $r = p_1/q_1$, and let $\Sigma_x = \cup_i x_i$ be a periodic orbit for a, so that $a(x_i) = x_{i+p_1}$. Let α^+ be defined by $\alpha^+(\theta) = a(x_{i+1})$ for $\theta \in (x_i, x_{i+1}]$, and let w^+ be obtained by replacing a with α^+. Then as in Theorem 3.2 w and w^+ have the same rotation

number, whereas w^+ takes Σ_x to itself, and therefore has a periodic orbit with period $\leq q_1$. ☐

4.2. **Rationality Conjectures.** In this section we state three conjectures on the rationality of R. These conjectures are related, and we explain how Conjecture 4.5 at least would follow if a certain dynamical problem (the *interval game*; see § 4.3) always had a positive solution.

Unfortunately, there are instances of the interval game which are unwinnable (i.e. unsolvable); however, it turns out that this dynamical problem *generically* has a positive solution, and that the exceptions must be quite special. This suggests a program to attack the rationality conjectures.

We adopt the following notational convention, which is consistent with our earlier use for positive w:

Definition 4.3. For arbitrary w and for real r, s, let $R(w, r-, s-)$ denote the supremum of $\mathrm{rot}^\sim(w)$ under all representations for which a and b are conjugate to rotations R_r and R_s respectively.

We would like to prove the following conjectures:

Conjecture 4.4. *Let w be arbitrary, and let $r, s \in \mathbb{Q}$. Then $R(w, r-, s-) \in \mathbb{Q}$.*

Conjecture 4.5. *Let w be arbitrary, and let $r, s \in \mathbb{Q}$. Then $R(w, r, s) \in \mathbb{Q}$.*

Conjecture 4.6. *Let $w \in [F, F]$. Then $\max_{r,s} R(w, r, s) \in \mathbb{Q}$.*

Although on the surface, Conjectures 4.4 and 4.5 seem very similar, the former very quickly reduces to the case of positive w:

Proposition 4.7. *Conjecture 4.4 is true if it is true for all positive w.*

Proof. Actually, the proof is a trick. Let $r = p_1/q_1$ and $s = p_2/q_2$. Then a^{q_1} is conjugate to the central element R_{p_1}, and similarly for b^{q_2}. It follows that for any factorization $w = w_1 w_2$ we have $w = w_1 w_2 = w_1 a^{q_1} w_2 z^{-p_1}$, and therefore we can write $w = w' z^{-N}$ for some sufficiently large integer N, where w' is positive, and $R(w, p_1/q_1-, p_2/q_2-) = R(w', p_1/q_1-, p_2/q_2-) - N$. ☐

It follows from Proposition 4.7 that $R(w, r-, s-)$ can be computed, at least for $r, s \in \mathbb{Q}$, by the method of § 3.8.

There are at least two classes of (r, s) for which $R(w, r-, s-)$ is known to be rational:

(1) at values of (r, s) where $R(w, \cdot, \cdot)$ is locally constant, we have $R(w, r-, s-) = R(w, r, s) \in \mathbb{Q}$; and

(2) at values of (r, s) where the upper bound in Theorem 3.24 is realized, we have $R(w, r-, s-) \in \mathbb{Q}$.

It is nevertheless true that $R(w, r-, s-)$ can be quite complicated.

Example 4.8. For any r, s (not necessarily rational) we have $R(aba^{-1}b^{-1}, r-, s-) = 0$. To see this, observe that $ba^{-1}b^{-1}$ is conjugate to R_{-r}, and therefore there is some point p for which $ba^{-1}b^{-1}(p) = p - r$; but then p is fixed by $aba^{-1}b^{-1}$, which therefore has rotation number 0.

Example 4.9. We now discuss $R(aba^{-1}b^{-1}, r, s)$, as an interesting counterpoint to Example 4.8. First of all, we claim that $R(aba^{-1}b^{-1}, r, s) = 0$ whenever r or s is

irrational. We argue analogously to the case of Example 4.8: suppose r is irrational, and let μ be an invariant probability measure for a supported on an exceptional minimal set. If there is an interval I with $\mu(I) = r$ and $\mu(b^{-1}(I)) < r$ then $aba^{-1}b^{-1}(I^-) < I^-$, so $\mathrm{rot}^\sim(aba^{-1}b^{-1}) \le 0$. But if $\mu(b^{-1}(I)) \ge \mu(I)$ for every interval I with $\mu(I) = r$ then b actually preserves μ, and therefore the action is semiconjugate to a linear action, and $\mathrm{rot}^\sim(aba^{-1}b^{-1}) = 0$.

If $r = p/q$ is rational, then $a(\cdot)$ has a periodic orbit $x_1, x_2, \cdots, x_q$ with indices corresponding to the cyclic order, and $a(x_i) = x_{i+p}$. If for some i, $x_j < b^{-1}(x_i) < x_{j+1}$ and $x_k < b^{-1}(x_{i-p}) < x_{k+1}$ with $k \le j - p - 1$ then $aba^{-1}b^{-1}(x_i) < x_i$ and $\mathrm{rot}^\sim(aba^{-1}b^{-1}) \le 0$. Otherwise we must have $k = j - p$, and $aba^{-1}b^{-1}(x_i) < x_{i+1}$, so that $\mathrm{rot}^\sim(aba^{-1}b^{-1}) \le 1/q$. If $s = p'/q$ we can build an action which is a q-fold cyclic cover of an action for which both a and b have fixed points; this shows that $R(aba^{-1}b^{-1}, p/q, p'/q) \ge 1/q$ and therefore $R(aba^{-1}b^{-1}, p/q, p'/q) = 1/q$.
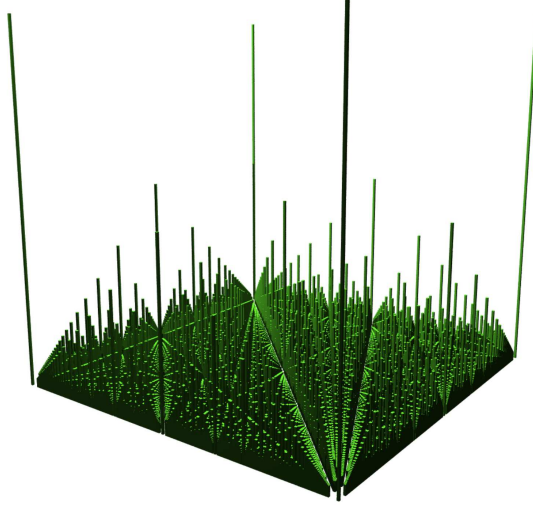
Compare with [20], Remark 3.8.

FIGURE 6. The "ziggurat" for $aba^{-1}b^{-1}$.

Note that Conjecture 4.4 can only fail for some w, r, s if (r, s) is slippery for w' (with notation from the proof of Proposition 4.7). Therefore Conjecture 4.4 is implied by the Slippery Conjecture.

4.3. **The Interval Game.** We pursue the following strategy to attack Conjecture 4.6, based more or less on the method of perturbation. This leads to a dynamical problem that we call *the interval game*. The structure of the set of "losing games" is very interesting, even when restricted to a very simple class of games (e.g. consisting entirely of rigid rotations).

The strategy is as follows. Given a word w, and given $r, s \in \mathbb{Q}$ we suppose that we have a representation for which a and b have the prescribed rotation numbers, and for which $\mathrm{rot}^\sim(w)$ is maximal. Suppose that $\mathrm{rot}^\sim(w)$ is irrational. We would like to adjust a to a new map a' (with the same rotation number as a) which will

adjust w to a new w' that has a strictly bigger rational rotation number. If we could do this, we would obtain a contradiction, and therefore conclude that $R(w, r, s)$ was rational after all.

We look for a suitable interval $I \subset S^1$ and require a' to agree with a outside I, but $a' > a$ on I. Providing I can be chosen in the complement of a periodic orbit, a and a' will have the same rotation number. The problem is that *increasing a on I* will *decrease a^{-1}* on $a(I)$, and it is not clear if we can find an adjustment for which the net effect on w is to increase its rotation number.

We abstract the situation in terms of a (one-player) game as follows.

Definition 4.10. An *interval game* consists of a finite collection of orientation-preserving homeomorphisms $\varphi_1, \cdots, \varphi_m$ (the *enemies*) and ψ. An interval $I \subset S^1$ *wins* if there is some positive integer n so that $\psi^i(I)$ is disjoint from $\varphi_j(I)$ for all j and for $0 \le i \le n$, and if $\psi^n(I^+)$ is contained in the interior of I, where I^+ denotes the rightmost point of I. The interval *loses* otherwise.

An interval game is *constrained* by a finite partition J of S^1 if the interval I must be chosen subject to being entirely contained in one of the intervals of J.

After conjugating w, we assume that w ends with a, and we let $w_1, w_2, \cdots, w_k$ be the (finitely many) suffixes of w that begin with a^{-1} (we take suffixes instead of prefixes because our group acts on the left).

Proposition 4.11. *Suppose the interval game has a winning interval, for $\varphi_i = w_i^{-1}$ and $\psi = w$, constrained relative to the partition consisting of intervals complementary to a finite orbit for a, and suppose $\mathrm{rot}^\sim(w)$ is irrational. Then $R(w, r, s) > \mathrm{rot}^\sim(w)$.*

Proof. Let I be a winning interval. We adjust a to a' on I, and consider the dynamics of a point in I under powers of w'. We can build a foliation encoding the dynamics on the mapping torus of w as follows. Let $C = \cup C_i$ be a cylinder with a vertical product foliation, decomposed into subcylinders each of which represents the dynamics of one letter of w. As we read w from right to left, the subcylinders from bottom to top represent the dynamics of each successive letter. Then the top of C is glued to the bottom by w.

Adjusting a to a' on I can be realized by adjusting the foliation in each sub-cylinder associated to an a or a^{-1} in w; see Figure 7. The figure shows the altered dynamics on an a-subcylinder and an a^{-1}-subcylinder.

Ignoring all a-subcylinders except the first (associated to the terminal a of w) for the moment, and under the hypothesis that I is winning, we see that the future itinerary of any point in I under powers of w' is the periodic orbit $w(I^+), w^2(I^+), \cdots, w^n(I^+), w(I^+)$; i.e. w' has rational rotation number, which is therefore *strictly* greater than that of w. Taking into account the adjusted dynamics at other a-subcylinders can only further increase the rotation number of w'. This completes the proof. $\square$

It is therefore important to understand precisely which interval games have a winning interval. First, we recall for the benefit of the reader, some elementary facts about the orbit of a point under successive powers of an irrational rotation. Fix some $\theta \in (0, 1)$, and a rigid rotation R_θ through θ. The case that $\theta \in \mathbb{Q}$ is completely straightforward, so suppose θ is irrational. We express θ as a continued
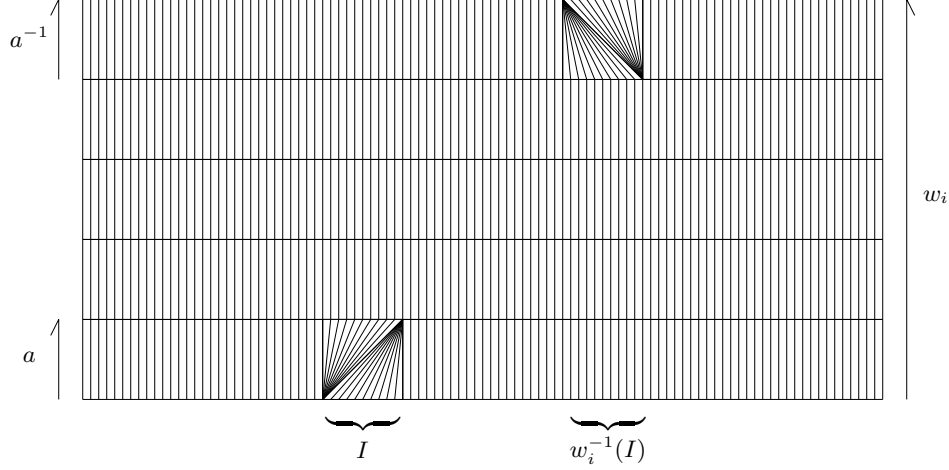
FIGURE 7. Adjusting the dynamics of a on I.

fraction

$$\theta = \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots$$

Define θ_i recursively by $\theta_0 = \theta$, $\theta_1 = 1 - a_1\theta_0$ and $\theta_{k+1} = \theta_{k-1} - a_{k+1}\theta_k$ for positive k. Fix $r \in S^1$, and let $r_i := R_\theta^i(r)$ denote the forward orbit.

The following lemma is straightforward (see e.g. [16] pp. 26–30):

Lemma 4.12. *Let r_{j_i} for $i = 1, 2, \cdots$ be the sequence of successively closest approaches to r; i.e. satisfying $|r - r_{j_i}| < |r - r_k|$ for $k < j_i$. Then the r_{j_i} alternately approach r from the left (if i is odd) and the right (if i is even), and $|r - r_{j_i}| = \theta_i$.*

Moreover, if r_a, r_b are adjacent elements of $\{r_i \mid 0 \leq i \leq n\}$ with $a < b$ then $|r_a - r_b| = |r - r_{b-a}|$; it follows that there are infinitely many odd i for which $|r - r_{j_i}| < |r_a - r_b|$ for all adjacent r_a, r_b with $a, b \neq 0$.

Definition 4.13. We say θ is *well approximated from the left* if there are a sequence of odd i for which $|\theta_i|/|\theta_{i-1}| \to 0$.

The set of $\theta \in S^1$ that are well approximated from the left has full measure; this is elementary, and follows e.g. by the kinds of estimates proved in [11].

The following theorem says that *generic* interval games (in a suitable sense) have a winning interval.

Theorem 4.14. *Consider the interval game associated to a collection $\varphi_1, \cdots, \varphi_m$ of C^1 diffeomorphisms, and ψ a rigid rotation through angle θ, where θ is irrational and well approximated from the left.*

Suppose there is a point $p \in S^1$ at which the derivatives of the φ_i are all different from 1. Then there is a winning interval I contained in any subinterval J sufficiently close to p.

Proof. Since the φ_i are all C^1, and since their derivatives at p are all different from 1, we can find an interval K contained in J so that the φ_i' are almost constant and bounded away from 1 on K. Intuitively, as we adjust the position of q near p, the

$\varphi_i(q)$ move almost linearly at speeds bounded away from 1. We can therefore adjust q to a location near p for which none of the $\varphi_i(q)$ are too close to some $R_\theta^i(q)$.

Let n be very large, and such that $R_\theta^n(p)$ approximates p from the left very well. We fix some very small ϵ, and require that

$$|p - R_\theta^n(p)| \leq \epsilon |R_\theta^i(p) - R_\theta^j(p)|$$

for all $0 < i, j < n$. Furthermore, for all $0 < i < n$ there should be some $0 < j < n$ not equal to i with $|R_\theta^i(p) - R_\theta^j(p)| \ll |K|$.

Because the φ_i' are bounded above and below and away from 1, for each i the set of $q \in K$ for which $|\varphi_i(q) - R_\theta^j(q)| \leq \max_{k \in K} |\varphi_i'(k)| \cdot |q - R_\theta^n(q)|$ for some j has measure of order $\epsilon |K|$. Choosing $\epsilon \ll 1/m$ we can find some q for which $|\varphi_i(q) - R_\theta^j(q)| > \max_{k \in K} |\varphi_i'(k)| \cdot |q - R_\theta^n(q)|$ for all i and all $0 < j < n$. Then $[R_\theta^n(q), q]$ is a winning interval.   □

A complete analysis of the interval game seems possible but difficult; however, we are able to completely understand the special case of a single enemy.

Theorem 4.15. *Consider the interval game with a single enemy φ, and suppose the rotation number of ψ is irrational. Let μ be an invariant probability measure for ψ. If φ does not preserve μ, there is a winning interval.*

Proof. First for simplicity we suppose that ψ is conjugate to a rigid rotation; equivalently, that μ has full support. The graph Γ of φ is a monotone $(1,1)$ curve in the torus $S^1 \times S^1$, and by hypothesis, it does not have slope 1 everywhere.

It follows that we can find a slope 1 curve L that locally supports Γ from above, and a point $(r, \varphi(r))$ which is locally the rightmost point of $\Gamma \cap L$; i.e. Γ is strictly below L in a neighborhood to the right of this point. Dynamically, φ is *strongly contracting* from the right at r; i.e. there is some ϵ so that for $|[r, s]| \leq \epsilon$, there is an inequality $|\varphi([r, s])| < |[r, s]|$. In fact, by the strictness of the inequality, there is an ϵ so that for all $\delta \leq \epsilon$ there is some $s(\delta) > r$ with $|\varphi([r, s(\delta)])| = |[r, s(\delta)]| - \delta$. Note that the smallest such $s(\delta)$ with this property has the additional property that $|\varphi([s, s(\delta)])| < |[s, s(\delta)]|$ for all $r < s < s(\delta)$; i.e. φ is strongly contracting from the left at $s(\delta)$. Geometrically, if we let L_δ denote the line with slope 1 obtained by translating L vertically down δ, then $(s(\delta), \varphi(s(\delta)))$ is the first time Γ crosses L_δ to the right of $(r, \varphi(r))$; see Figure 8.

We adopt the notation $r_i := \psi^i(r)$. Since $\mathrm{rot}(\psi)$ is irrational, by Lemma 4.12 we can find an arbitrarily big m so that r_m is a closest approach to r from the left; in particular, $|r - r_m| < |r_a - r_b|$ for all $0 < a, b < m$. Let $u = |[r_m, r]|$.

Let r_a, r_b, r_c, r_d be successive orbits such that $r_c \leq \varphi(r) < r_d$. Let $t_1 = |[r_a, r_b]|$ and $t_2 = |[r_b, r_c]|$, and $v = |[r_c, \varphi(r)]|$. Note that $t_1 > u$ by the definition of u. We can assume (by taking m sufficiently big) that $t_2 + v \ll \epsilon$, and therefore there is a point $s(t_2 + v)$ to the right of r with properties as above.

Let $t = |[r, s(t_2 + v)]|$. Then $\varphi(r + t) = r_b + t$, and $|\varphi([r_m + t, r + t])| \leq u < t_1$, so $r_a + t < \varphi(r_m + t) < r_b + t = \varphi(r + t)$. In particular, an interval of the form $[r_m + t - \delta, r + t]$ is winning, for sufficiently small δ.

It remains to consider the general case where μ does not have full support. Actually, the argument in this case is essentially the same as that above. In place of Γ we can consider the curve $\Gamma' := \{(\int_0^r d\mu, \int_0^{\varphi(r)} d\mu)\}$. This "graph" might have horizontal and vertical segments, but otherwise we can use the same argument with curves L, L_δ applied to Γ' in place of Γ.   □
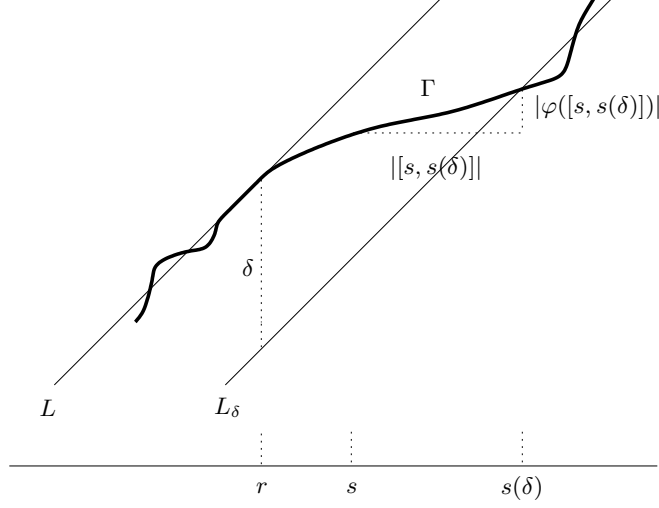
FIGURE 8. φ is strongly contracting to the right of r.

Interestingly enough, the exceptional case that φ and ψ are both rigid rotations (after a semiconjugacy) turns out to be nontrivial:

Theorem 4.16. *Consider the interval game in which both the single enemy φ and ψ are rigid rotations through u, t respectively. Then there is a winning interval if and only if (t, u) is contained in an explicit open subset U of the unit square described below (see Figure 9).*

Proof. Winning (t, u) are classified by which iterate n of ψ certifies the win, and the smallest integer m such that $nt < m$. Let $U(m)$ be the subset of U with a given value of m. Then one sees directly that $U(1)$ is the union over all $n \geq 2$ and $1 \leq i \leq n - 1$ of the interior of the triangles with vertices $(\frac{1}{n}, \frac{i}{n})$, $(\frac{1}{n-1}, \frac{i-1}{n-1})$, and $(\frac{1}{n-1}, \frac{i}{n-1})$.

On the other hand, by rescaling the interval $[0, m]$ by a factor of $\frac{1}{m}$ it is clear that if $(\frac{t}{m}, \frac{u+i}{m}) \in U$ for all $0 \leq i < m$ then $(t, u) \in U$, and any element of $U(m)$ is of this kind. This gives a recursive description of U. $\square$

The complement of U is the attractor of the IFS generated by the set of projective linear transformations of the form

$$(x, y) \rightarrow \left(\frac{x + n}{x + n + 1}, \frac{y}{x + n + 1} \right), \quad (x, y) \rightarrow \left(\frac{x + n}{x + n + 1}, \frac{x + y + n}{x + n + 1} \right)$$

for all non-negative integers n.

Theorems 4.15 and 4.16 together present an essentially complete picture of the interval game with a single enemy.

Notice for every rational u the set of t with (t, u) in U is an open, dense subset of $[0, 1]$. As a corollary we get strong constraints on $R(w, r, s)$ for w containing at most one a^{-1}:

Corollary 4.17. *Suppose w is a word of the form $va^{-1}b^n a$ for some v containing no a^{-1}, and suppose $r, s \in \mathbb{Q}$. If $R(w, r, s)$ is irrational then $(R(w, r, s), ns)$ is not in U.*

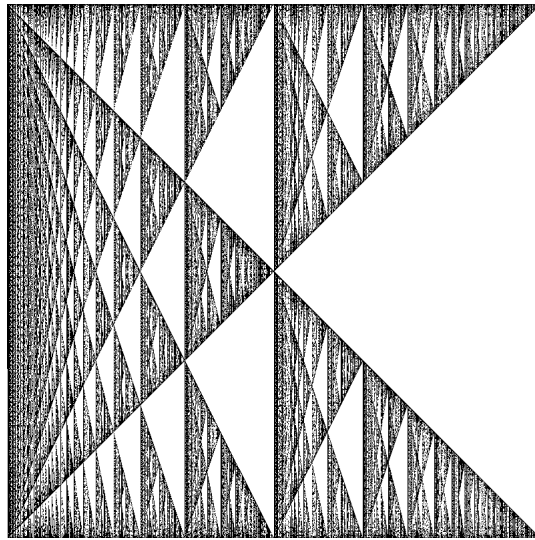*In particular, there is a dense G_δ subset of $[0, 1]$ that $R(w, r, s)$ avoids.*

FIGURE 9. The subset U (in white) of the unit square shows the winning interval games for a pair of rigid rotations.

Remark 4.18. For each rational u the set of t with (t, u) not in U is the attractor (i.e. the limit set) of an explicit finitely generated subsemigroup of $\mathrm{SL}(2, \mathbb{Z})$ acting projectively on the interval. For example, if $u = 1/2$, the set of "bad" t is the limit set of the semigroup generated by the matrices

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

If T is a finitely generated semigroup of contractions of the interval whose images are disjoint, and some of the maps have a neutral (i.e. parabolic) fixed point, Urbański ([21]; see also [18]) showed, generalizing work of Bowen [1], that the Hausdorff dimension of the limit set is the least zero of the pressure function P, defined by the formula

$$P(s) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{M \in T_n} \|M'\|^s$$

where T_n is the subset of T consisting of words of length n, and $\|\cdot\|$ is the supremum norm. Actually computing this dimension in practice seems hard.

5. ACKNOWLEDGMENTS

APPENDIX A. HIGHER RANK

For convenience, we state the analogues of our main theorems to higher rank. The proofs of these theorems are routine generalizations of the proofs in the body of the paper, and are omitted.

Throughout this section, let F be a free group of rank n with free generating set $a_1, a_2, \cdots, a_n$. For real numbers $r_1, r_2, \cdots, r_n$ and $w \in F$, let $R(w, r_1, r_2, \cdots, r_n)$ denote the maximum of $\mathrm{rot}^{\sim}(w)$ under representations $F \to \mathrm{Homeo}^+(S^1)^{\sim}$ for which $\mathrm{rot}^{\sim}(a_i) = r_i$.

Theorem A.1 (High rank Rationality Theorem). *Suppose w is positive. If the r_i are all rational, so is $R(w, r_1, \cdots, r_n)$. Moreover, if w has at least one a_i, the denominator of $R(w, r_1, \cdots, r_n)$ is no bigger than that of r_i.*

Theorem A.2 (High rank Stability Theorem). *Suppose w is positive. Then R is locally constant from the right at rational points; i.e. for every collection of rational numbers r_i, there is an $\epsilon > 0$ so that $R(w, \cdots)$ is constant on $[r_1, r_1 + \epsilon) \times \cdots \times [r_n, r_n + \epsilon)$.*

Conversely, if $R(w, r_1, \cdots, r_n) = p/q$ (where p/q is reduced) and the biggest power of consecutive a_is in w is $a_i^{m_i}$ then there is an inequality

$$R(w, r_1, \cdots, r_i + 1/m_i q, \cdots, r_n) \geq p/q + 1/q^2$$

Theorem A.3 (High rank Stairstep Theorem). *Let w be positive, and suppose we are given rational numbers p_j/q_j for $j \neq i$ and c/d so that*

$$R(w, p_1/q_1, \cdots, t_i, \cdots, p_n/q_n) = c/d$$

for some real t_i (so necessarily $d \leq q_j$ for each j). Then the infimum t_i with this property is rational, and there is an algorithm to compute it. Moreover, if p_i/q_i is this infimal value, $R(w, p_1/q_1, \cdots, p_n/q_n) = c/d$.

Theorem A.4 (High rank Isobar Theorem). *Let w be positive. For any rational p/q the set of $(r_1, \cdots, r_n) \in [0, 1]^n$ such that $R(w, r_1, \cdots, r_n) \geq p/q$ is a finite sided rational polyhedron, whose boundary consists of finitely many polyhedra on which at least one r_i is constant.*

The results in § 4 also generalize in a straightforward way to higher rank, but we omit the statements.

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